Fast Price Discovery*

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Abstract

Allocating goods efficiently can take a long time. That time spent is costly to participants. We study mechanism design in a symmetric, IPV, heterogeneous item allocation environment where the designer's objective is a flexible function of the allocative efficiency and the time taken to complete a mechanism. Assuming that the size of action sets is sufficiently small, a mechanism offering participants a sequence of potential clearing prices is optimal. We show how activity rules, such as the Milgrom-Wilson activity rule in the FCC spectrum auctions, help speed up auctions: They prevent the formation of sub-optimal equilibria. When the number of bidders is sufficiently large and the value distribution for any bundle is sufficiently thin-tailed, a descending auction is the optimal way to allocate goods. An equivalence result between the designer's speed objective and a complexity objective based on the number of possible actions is also discussed.

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1 Introduction

In mechanism design, an important question is how to allocate goods to maximize welfare when the participants' values are not known ex-ante. Theoretically, the celebrated Vickrey–Clarke–Groves (VCG) mechanism provides a general solution on how to solicit values and allocate goods to maximize welfare. However, VCG mechanisms are rarely used in practice. This is for a variety of reasons, including that when L items are being allocated, participants in the mechanism have to formulate and state valuations for all 2^L different bundles they could acquire.[1] Indeed, most allocation mechanisms seen in practice are dynamic. This is true even when there is only a single item for sale — compare the number of Dutch or English auctions that take place versus sealed-bid auctions.

Dynamics make it much easier for participants because they face simpler decisions than they would in a static mechanism. In particular, bidders in a dynamic mechanism may have to perform less contingent reasoning in order to decide their actions (Li, 2017) as well as having fewer possible actions to consider taking at any given history (Nagel and Saitto, 2023). Overall, the number of preferences that need to be stated in order achieve the efficient allocation is dramatically less than the 2^L needed in the VCG mechanism.

While dynamic auctions can make it simpler for agents to make decisions in each round, the number of actions agents need to take can be very salient and large. For example, the longest FCC spectrum auctions took over seven months and on average took almost a month and about 74 rounds to complete (Table 1). Bidders were required by the Milgron-Wilson activity rule to submit bids almost every round in order to be eligible to continue bidding, which means these long auctions imposed large process costs. Other examples where the speed of allocation matters include commodities, which spoil quickly, and electricity, where demand is imminent and blackouts are extremely undesirable.

	Mean	Standard Deviation	Median	Min	25%	75%	Max
Duration (Days)	27	37	14	1	7	29	227
Number of Rounds	74	71	47	1	27	94	341

Table 1: Summary statistics for completion time of FCC spectrum auctions ($N = 99$).

In this paper, we study properties efficient (welfare-maximizing) dynamic auctions when there is a cost for the auction taking a long time. Our formal environment is a general IPV, heterogeneous goods problem: There are L items to sell and N bidders. The bidders have symmetric, independent private values over bundles drawn from commonly known distribution F and are quasi-linear with respect to transfers paid. The seller has no value for the items and only values the transfers. We suppose that the mechanism limits the amount of information a bidder can communicate at any given time to κ bits. There are many possible reasons for why there would be a restriction on the amount of information a bidder can communicate at one time. For example, there could be psychological constraints on the number of choices a bidder can consider at once or technological/physical constraints on what the

[1]With assumptions about preferences, the number of preferences that must be stated can be reduced, but a large reduction in the number of preferences that must be stated requires strong assumptions on preferences. See Nisan (2006) for a survey.

bidding system can process at any given time.[2] We assume that the mechanism designer has an increasing function c assigning a "price of time" for every additional (non-simultaneous) action that is taken. The designer wants a mechanism that maximizes allocative efficiency less this price of time. We refer to such mechanisms as *c-efficient*.

We show that adding this cost of time explains a number of commonly seen phenomenon. First, a mechanism soliciting optimal consumption bundles at a sequence of prices will always be c-efficient. This provides a justification as to why almost all dynamic allocation mechanisms seen in practice are *price discovery mechanisms*. Second, we formalize the use of activity rules: They refine away sub-optimal equilibria. In particular, price discovery mechanisms have multiple equilibria and all equilibria which are non-truthful[3] are less c-efficient than the truth-telling equilibrium. Any activity rule which does not prevent truth-telling behavior will only remove sub-optimal equilibria and furthermore the more an activity rule restricts behavior up to that point, the more inefficient equilibria will be refined away. Third, we formalize the commonly given anecdote on why descending auctions are used: They allocate goods quickly (Mishra and Parkes, 2009). When there are a sufficient number of participants and the value distribution for any bundle has thin enough tails, the c-efficient auction will be a descending auction.

In Section 1.1, we give a more detailed summary of our results. In Section 1.2, we explore the related literature. Section 2 gives the formal model and desideratum. Section 3 explores price discovery mechanisms and Section 4 explores the effects of activity rules. Section 5 characterizes the descending auction as c-efficient when there are enough bidders. In Section 6, we show a connection between speed and complexity. Finally, in our discussion section (Section 7) we give a behavioral application and consider a worst-case analysis.

1.1 Summary of Results

Price Discovery. Our first main result is that when κ is a divisor of L, e.g., the bandwidth is equal to the number of items, all c-efficient mechanisms are allocation equivalent to a mechanism which proposes a series of prices where bidders state a consumption bundle (Theorem 3.2). In the theorem, we also prove that such a mechanism has a truth-telling equilibrium. In the multi-item case, we assume that preferences satisfy gross substitutes (Assumption 2.1), so that there exists a price that clears the market. There are two main difficulties in proving this theorem. First, finding transfer functions that make truth-telling consumption bundles an equilibrium strategy. Second, proving that querying consumption bundles at each price is optimal; *prima facie*, there may be more efficient ways of conditioning on a sequence of κ bits.

Activity Rules. Our next main result shows a formal justification for activity rules. An activity rule is any restriction on a price discovery mechanism that limits a bidder's behavior based on her behavior in previous rounds. We consider a class of non-truthful equilibria we

[2] A limit on the number of bits of information someone can process at once is equivalent to a limited channel capacity, which is the foundation for many rational inattention models. See Maćkowiak et al. (2023) for a survey of rational inattention models.

[3] Straightforward is another common term in the literature for truthful reporting of consumption bundles at a sequence of prices.

call *lazy equilibria* where bidders either submit the same message multiple rounds in a row or report truthfully. We show there exist lazy equilibria that an activity rule will make non-rationalizable. We prove that all lazy equilibria of the c-efficient dynamic price game are less c-efficient than the truth-telling equilibrium (Theorem 4.2). This provides a justification for activity rules: Any restriction on bidders' behavior that rules out behavior inconsistent with gross substitute preferences refines away "slow" equilibria.

Descending Auctions. The c-efficient mechanism wants to minimize the expected number of steps in the mechanism. One force to consider when designing such a mechanism is to propose high probability clearing prices early. This force dominates when the number of bidders is sufficiently large: When the prior value distribution over any bundle is sufficiently thin tailed and there are enough bidders, the c-efficient mechanism is a descending (Dutch) auction (Theorem 5.2). In this case, sufficiently thin-tailed means that the hazard hate increases super-exponentially; this condition means that the value distribution has tails thinner than Gumbel distributions. The proof of this theorem shows that although the optimal maximum price queried increases as the number of bidders increases, it increases sufficiently slow such that the probability of the clearing being at said price increases.

Note that even when the value distribution is bounded, this result is stronger than the approximate optimality of posted prices (Wilson, 1977) as the number of bidders grow. Proposition 5.3 shows that there is a gap in terms of the approximation: We can find scenarios where the descending auction is ex-post allocatively efficient, but the optimal posted price obtains no more than $\frac{3}{4}$ of the maximum ex-ante efficiency. In the Appendix, we can also characterize general c-efficient mechanisms: It is without loss to restrict to *divide-and-conquer auctions* (Definition D.1 and Theorem D.2).

Connections to Complexity. In Section 6, we connect fast price discovery to different notions of complexity. Corollary 6.4 shows an equivalence result between menu costs and a price of time. Menu costs are defined as it being costly for participants to choose among a large set of choices. We also show that efficient mechanisms will offer participants a series of binary choices if their disutility for menus of size k is bounded below by $\log k$ (Proposition 6.2). In addition to our auction results, in Section 7, we also apply this result to the classic experimental design question of how to truthfully elicit willingness-to-pay (WTP) from subjects. Our model provides a theoretical argument that the common Becker et al. (1964) method for soliciting WTP via a series of binary choices minimizes menu cost complexity. Finally, we connect to the prior literature on complexity and show that supplementing the notion of complexity found in Nagel and Saitto (2023) with a lexographically weaker distaste for taking a large number of actions will also derive the results found in this paper.

Other Results. Observe that there is a tension between finding the efficient allocation and the number of actions that each participant must take. Our next result proves that for any c-efficient mechanism where preferences generate a value distribution that is sub-exponential (Definition 2.7), it is without loss to assume that each participant takes a finite number of actions (Lemma 2.8). A finite number of steps means the market may only approximately clear and may not achieve first-best allocative efficiency. This lemma provides a technical innovation upon the previous extensive-form market design literature by allowing a tractable approach to designing extensive-form mechanisms when the value distribution is continuous and unbounded.

In Section 7, we consider c-efficient mechanisms under the worst-case realization of bidders' value distribution. We propose the *simultaneous binary auction* which performs a binary search over clearing prices for each item independently. This mechanisms is much faster than, for example, the ascending or descending auction. If there are M possible clearing prices for each of the L goods, then the simultaneous binary auction achieves the asymptotically optimal rate of $O(\log M)$ queries, whereas the ascending and descending auctions have $O(L \cdot M)$ queries (Proposition 7.3). This implies that especially when the number of items is relatively large compared to the number of bidders, the ascending auction is far slower than an efficient allocation process needs to be.

1.2 Related Work

Our work is primarily related to two strands of literature: process considerations in mechanism design and price discovery or tâtonnement processes. In mechanism design, the most common process consideration has been simplicity. Simplicity has often been captured by a notion of "strategy-proofness" or dominant strategies (Vickrey, 1961). This means that each participant has a strategy that is weakly optimal for them no matter what other participants choose to do. Recently, there have been a number of papers strengthening dominant strategies to notions that require less reasoning about what other players are going to do (Li, 2017; Pycia and Troyan, 2023). These definitions are all focused on contingent reasoning limitations: what is "obvious" for players to want to do when they have limited ability to reason about what actions other participants (or their future selves) might take.[4] Nagel and Saitto (2023) explores a notion of complexity (roughly) based on the largest choice set that must be considered and find that many price discovery mechanisms have minimal complexity under their definition. Our work not only provides a different justification for price discovery mechanisms, speed, it also provides a desideratum for picking among them. In Section 7, we show how speed and simplicity are closely linked. Other recent papers such Akbarpour and Li (2020) have process considerations in dynamic mechanisms that are neither strategic complexity nor speed. Like those papers, we also speak to the importance of dynamic considerations in market design. None of the papers above explore the general, heterogeneous allocation problem like our work here does.

Papers such as Nisan and Segal (2006) and Segal (2007) demonstrate that clearing prices must be found for any efficient equilibrium and papers such as Milgrom (2000) and Ausubel (2006) show that such prices can be found via dynamic auctions, even when there is incomplete information. Milgrom (2009) explores succinct ways to express preferences via linear program constraints, but does not study dynamic formats. Blumrosen et al. (2007) and Kos (2012) consider limiting the message space, but are focused on the single-item case and outcomes only, not characterizing the dynamic mechanism. Our work complements previous

[4] Moving from a Bayesian-Nash equilibrium to an ex-post Nash equilibrium to a dominant strategy equilibrium are also all weakening of the amount of contingent reasoning that must take place.

research by exploring the efficient way to find prices, not just showing information theoretic bounds on finding such prices. More distantly, we relate to papers exploring price discovery processes that also have large fluctuations in "prices," but in other settings (e.g., Ashlagi et al. (2022)). We also relate to papers such as Komo et al. (2024) by providing a justification for the Dutch auction. They focus on single-item auctions and prove that Dutch auctions are uniquely suited to prevent shill bidding while we consider the multi-item setting and use speed as our desideratum.

We also speak to the study of activity rules in auctions. There is a small body of literature studying activity rules, such as Ausubel and Baranov (2020). However, these papers focus on strategic reasons that bidders may want to delay bidding, and attempt to explicitly design activity rules that allow participants to express preferences satisfying particular axioms. Here, we instead take a more abstract view of activity rules and instead of explicitly designing activity rules with specific desiderata in mind, we try to formally understand how activity rules can improve the functioning of an auction.

2 Model

2.1 Basic Set-up

We consider the heterogeneous goods allocation problem with symmetric, independent private values. There are L items, indexed by $\ell \in \{1, \ldots, L\}$, to be allocated among N bidders, indexed by $i \in \{1, \ldots, N\}$. Each bidder has type $\theta_i \sim F \in \Delta(\Theta)$ i.i.d. We assume each buyer i has quasi-linear utility when she is allocated some bundle $x_i \in \{0,1\}^L$ and pays transfer t_i:

$$u_i(x_i, t_i, \theta_i) = v(x_i, \theta_i) - t_i.$$

We will use x, t and θ to denote the vectors of the respective values. There is also seller who has zero value for the goods: $u_0 = \sum_{i=1}^{N} t_i$. The seller does not participate in the mechanism, but is considered when computing social welfare.

We make the standard assumption in the literature that preferences satisfy gross substitutes. Gross substitutes mean that if the price of good ℓ increases and the price of good ℓ' stays the same, the demand for ℓ' must weakly increase. Formally, let $X_i^*(p, \theta_i) \in \operatorname{argmax}_{x_i \in \{0,1\}^L} \{v(x_i, \theta_i) - p \cdot x_i\}$ be bidder i's demand when her realized type is θ_i and the price vector is p. Then,

Assumption 2.1. Consider any bidder i, type θ_i and price vectors p and p' such that $p' \geq p$. We assume preferences satisfy **gross substitutes**: For all ℓ such that $p'_\ell = p_\ell$, it is the case that there exists X^* such that $X^*_{i,\ell}(p', \theta_i) \geq X^*_{i,\ell}(p, \theta_i)$.

As notation, we will let $\overline{X}^*(p, \theta) \equiv \sum_i X^*(p, \theta)$ be defined as the aggregate demand when the price is p and the realized type vector is θ. As documented by Gul and Stacchetti (1999) and Milgrom (2000), assuming gross substitutes guarantees that a **clearing price** $p^*(\theta)$ exists: $\overline{X}^*(p^*(\theta), \theta) = \mathbf{1}^L$. This assumption is commonly seen in the literature, e.g., Ausubel (2006). Observe that when a clearing price exists, the allocation it induces will be efficient. Further, when preferences are such that clearing prices always exist, any efficient allocation will be supported by a clearing price (Nisan and Segal, 2006).

The designer is free to design any extensive-form game, but said game cannot condition directly on bidders' types. Bidders' types can only affect outcomes through the actions bidders takes. In the following subsections, we will layout restrictions on the designer's objective function and choice in mechanisms. Let $\sigma_i(\theta)$ be a pure strategy for buyer i conditional on the realized type θ and let $\sigma(\theta)$ be the entire strategy profile. We will be interested in the following three equilibrium concepts:

Definition 2.2. A strategy profile σ is a **Bayes incentive compatible** (BIC) equilibrium when for all i, θ_i, and θ_i',

$$\mathbb{E}_{\theta_{-i}}\left[u_i(\sigma(\theta), \theta_i)\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i(\sigma(\theta_i', \theta_{-i}), \theta_i)\right].[5]$$

Definition 2.3. A strategy profile σ is an **ex-post** (XP) equilibrium when for all i, θ, and σ_i',

$$u_i(\sigma(\theta), \theta_i) \geq u_i((\sigma_i', \sigma_{-i}(\theta)), \theta_i).$$

Definition 2.4. A strategy profile σ is a **dominant strategy** (DS) equilibrium when for all i, θ, and σ',

$$u_i(\sigma(\theta), \theta_i) \geq u_i(\sigma', \theta_i).$$

BIC equilibria are Nash equilibria where allowed deviations are to playing like some other type. XP equilibria are Nash equilibria when the realized type vector θ is common knowledge. DS equilibria are equilibria where playing the equilibrium strategy is weakly optimal, no matter how the other bidders behave. If an equilibrium is DS, then it is XP, which in turn implies BIC. As notation, let $\Sigma_{BIC} \supseteq \Sigma_{XP} \supseteq \Sigma_{DS}$ be the set of strategies that form a BIC, XP, and DS equilibrium, respectively. Note that we deviate from standard mechanism design in that we focus on indirect mechanisms instead of direct mechanisms which means that $\Sigma_{DS} \neq \Sigma_{XP}$. However, as is standard in mechanism design, we focus on partial implementation and allow the designer to select the equilibrium. Let us define a **mechanism** as (x, t, σ) where, for message space or action space domain $\mathcal{M}$, $x : \mathcal{M} \to \Delta^N\left(\{0,1\}^L\right)$[6] is the allocation function, $t : \mathcal{M} \to \mathbb{R}^N$ is the transfer function, and $\sigma \in \Sigma_{BIC}$ is the equilibrium strategy.

2.2 Desiderata

Our primary focus in this paper is on speed: We want mechanisms that terminate quickly (small number of steps). The goal will be to maximize total welfare (including the seller's utility) less some cost for each additional step in the mechanism.

Formally, consider a sequence of non-simultaneous actions $y = \{y_k\}_{k=1}^{\infty}$. We say the mechanism terminates after K steps if the outcome is invariant to entries after the first K: If the mechanism has terminated after K steps, then for sequences y and y' such that $y_k = y_k'$ for all $k \leq K$, it is the case that $(x, t)(y) = (x, t)(y')$. Further, there exists y'' such that $y_k = y_k''$ for all $k < K$ and $(x, t)(y) \neq (x, t)(y'')$. We define $K^*(x, t, \sigma, \theta)$ as the number of steps it takes the mechanism (x, t, σ) to terminate when realized types are θ.

[5] $u_i(\sigma(\theta))$ represents the outcome (x, t) realized when σ is played and the realized type is θ.

[6] $\Delta^N(\{0,1\}^L) \equiv \left\{y : y \in [0,1]^{L\times N}, \forall \ell : \sum_{i=1}^N y_{\ell,i} = 1\right\}$, i.e., the N-simplex over L goods.

We define the function $c : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as a "cost of time" which parameterizes the trade-off between obtaining additional information about bidders' values and the cost of queries.

Definition 2.5. A mechanism (x, t, σ) is c**-efficient** under F if

$$(x, t, \sigma) \in \underset{\substack{x':\mathcal{M}\to\Delta^N\left(\{0,1\}^L\right),\\ t':\mathcal{M}\to\mathbb{R}^N,\\ \sigma'\in\Sigma_{BIC}}}{\operatorname{argmax}} \mathbb{E}_{\theta_i\sim F}\left[\sum_{i=1}^{N} v(x_i'(\sigma'(\theta)), \theta_i) - c\left(K^*(x', t', \sigma', \theta)\right)\right] \tag{1}$$

We assume that the cost of time c is strictly monotone, i.e., the designer strictly prefers the mechanism to finish quickly, all else equal. The functional form of c in Equation (1) flexibly includes many design considerations. For example, if the auction must finish within a specific time frame, we can take $c(\overline{K})$ to be sufficiently large in order to guarantee that the c-efficient mechanism finishes in fewer than $\overline{K}$ steps. If the marginal cost of time is constant, then $c(K) = \underline{c} \cdot K$. If the type space is finite, Equation (1) also allows for lexographic preferences: We can suppose that $\max_{K\leq|\Theta|}(c(K+1)-c(K)) < \min_{\theta\in\Theta, x\in\Delta^N(\{0,1\}^L)} v(x,\theta)$ in order to endow the designer with lexographic preferences. We can also reverse the ordering by taking $c(1)$ sufficiently large; in this case any arbitrary allocation of at most one good per person is optimal. The key economic assumption baked into Equation (1) is that the designer's preferences over allocations is additively separable from her time costs; if, for example, the designer cared about time costs differently depending on who got allocated the goods, then our assumption would be violated.

2.3 Limited Bandwidth Mechanisms

Without restrictions on the action space, a c-efficient mechanism is a static VCG mechanism (or lottery if $c(1)$ is large enough). However, we know that running VCG auctions is usually not practical. As mentioned in the introduction, one reason as to why not is that expressing preferences over all bundles is too burdensome. We formalize this by placing a limited bandwidth on the amount of information that can be transmitted to the mechanism at once. This is exactly analogous to the channel capacity limits for processing information frequently seen in the rational inattention literature.[7] A κ-bandwidth mechanism is defined as a mechanism where each bidder can only communicate κ bits in each step. The formal assumption is below.

Assumption 2.6. Let $\mathcal{M}_\kappa = \left\{\{0,1\}_k^{\kappa\times N}\right\}_{k=1}^{\infty}$ be the space of all sequences of N bidders submitting κ-bit messages. All mechanisms considered are κ**-bandwidth mechanisms**: The mechanism's domain is restricted to $\mathcal{M}_\kappa$.

A κ-bandwidth mechanism is equivalent to assuming that participants have a κ bit channel capacity in order to process information. If a bidder can only process a limited amount of

[7] Instead of restricting information based on some information theoretic function, we instead restrict the message space.

information per time step, the c-efficient mechanism will provide only the most pertinent information in that time step. In Section 6, we will see that certain complexity considerations will yield similar restrictions to the κ-bandwidth mechanism.

Observe that it is without loss with respect to outcomes to restrict to κ-bandwidth mechanisms for $\kappa > 0$. This is because each bidder could encode her value for each of the 2^L bundles using an infinite sequence of κ bits. Further observe that most dynamic auctions seen in practice are κ-bandwidth mechanisms with κ relatively small. For example, any auction where bidders bid quantities at a series of prices is a ($\kappa = L$)-bandwidth mechanism. This category includes the FCC auctions, as bidders choose to continue or drop out of each license as the price increases. This category also includes all simultaneous or sequential ascending or descending auctions as are used in some timber and electricity markets. Other examples of κ-bandwidth mechanisms include auctions where bids are prices at a series of quantities and the number of possible prices to bid is constrained. κ-bandwidth mechanisms also can incorporate auctions with entrance fees. Further, even for dynamic auctions that are not κ-bandwidth mechanisms for κ small, most of these auctions are hybrid auctions where there is a single static round of bidding either before or after the dynamic stage. The dynamic stage is normally a κ-bandwidth mechanism. Our results will also speak to how to complete the dynamic portion of these hybrid auctions.

2.4 Finite Termination

In much of the mechanism design literature, continuous, potentially unbounded value distributions are used. However, in the extensive-form market design literature, finite type spaces are often assumed because designing an auction with an infinite number of leaves is usually intractable. In our setting, the cost of time c naturally "coarsens" the type space allowing us to bridge from continuous type spaces to finite extensive-form games.

Before describing the conditions under which the c-efficient mechanism will terminate in finite type when the type space is infinite, let us observe that if the type space is finite, the c-efficient mechanism will always terminate in finite time. To see this just observe that each bidder can send $\lceil 2^L/\kappa \cdot |\Theta| \rceil$ messages fully encoding their value and then the mechanism can run VCG to obtain allocative efficiency. Thus, no c-efficient mechanism will run longer than that number of steps.

In order to have a c-efficient mechanism always terminate in finite time, there must be thin enough tails in the value distribution and the cost function must increase super-linearly with the number of steps. To see why this assumption on the value distribution is sufficient, consider a single-item auction and note that the value of querying a price is distinguishing the highest value from all other values. If the distribution has fat tails, for any truncation, the expected difference between the first and second order statistic may be bounded away from 0. Thus, if c is sufficiently small and if multiple people have a value for a good above some price, it will be optimal to query another price. To rule out all such examples, a sufficient condition is that the distribution of valuations is sub-exponential:

Definition 2.7. Let F_x be the cdf of the distribution of $v(x, \theta_i)$ given that $\theta_i \sim F$. Preferences are **sub-exponential** with bound $y^* : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ if for all x, F_x admits a pdf f_x and for all $\gamma > 0$ and $y > y^*(\gamma)$, it is case that $\frac{1-F_x(y)}{f_x(y)} \leq \gamma$.

This is the standard definition of sub-exponential, but applied to each possible bundle separately. The sub-exponential assumption is commonly used in the literature, see, e.g., Ferreira and Weinberg (2020).[8]

To see why the assumption on the cost function is sufficient, observe that when c is super-linear, there exists a $\underline{c}$ such that the marginal cost of another time step is at least $\underline{c}$. Thus, values that are within $\underline{c}/2$ will never be distinguished from each other as the marginal value of distinguishing will be less than the marginal cost of distinguishing.

Lemma 2.8. *Suppose preferences are sub-exponential with bound y^* and there exists $\underline{c} > 0$ such that for all K, $c(K+1) - c(K) \geq \underline{c}$. If a mechanism (x, t, σ) is c-efficient, there exists $\overline{K}(y^*) < \infty$ such that for all θ, $K^*(x, t, \sigma, \theta) < \overline{K}(y^*)$.*

The proof of Lemma 2.8, and all other results can be found in the Appendix. To see what can occur without sub-exponential preferences, consider a single-item auction and suppose values are drawn from an exponential distribution with rate λ. For any value y, suppose at least two bidders have value above y (such an event has positive probability for all y). If the good is randomly allocated between the two bidders, then the expected valuation is $y + \frac{3}{4\lambda}$. On the other hand, if the good was allocated to the bidder with the highest valuation, the expected value would be $y + \frac{3}{2\lambda}$. Thus, if $\frac{1}{\lambda}$ is sufficiently large compared to the marginal c, it will always be c-efficient to try to query another value and distinguish the two bidders from each other. More abstractly, the "memory-less" property of the exponential distribution will cause there to not be any uniform bound on $K^*(x, t, \sigma, \theta)$. Likewise to see what occurs without super-linear costs, let us consider strictly sub-linear costs for a single-item auction. In this case, distinguishing between the highest two values could strictly increase the designer's objective as the difference between the first and second moment could be linear in the potential support whereas the cost is sub-linear.

Having the mechanism always terminate in a finite number of steps will be convenient for our future results so that we do not have to concern ourselves with infinite-depth game trees.

Assumption 2.9. If $|\Theta| = \infty$, preferences are sub-exponential with bound y^* and there exists $\underline{c} > 0$ such that for all K, $c(K+1) - c(K) \geq \underline{c}$.

3 Price Discovery and Limited Bandwidth Mechanisms

In this section, we show that a c-efficient, κ-bandwidth mechanisms is a price discovery mechanism for small κ: Bidders truthfully report their optimal consumption bundles at a series of prices. We call this implementation a *dynamic price game*. There are two main goals we work towards: (1) proving that the designer can incentive truthful reporting of consumption bundles at a sequence of prices (in an ex-post manner) and (2) proving that such mechanisms are in fact optimal in the class of all κ-bandwidth mechanisms.

Towards the first point, the key problem is to design a transfer scheme that incentivizes truthful reporting at different prices. We can examine the VCG mechanism for intuition on what is necessary to incentivize truth-telling. First, conditional on being allocated an item,

[8] They in fact use the stronger condition that $y^* = o(1/\gamma)$.

the price the winner pays cannot depend on the winner's own value for that item. Second, the more an item's losers value the item, the more the winner must pay. Combining these two points, we design a transfer scheme that "credits" or "debits" the eventual winner's payment for an item based on how the other bidders' demand changes between prices. If more than two people demand an item at some price, then the transfer "debits" this price. But if the price is then increased so that there is no longer over-demand for the item, the winner is "credited" from the decrease in demand from the price increase. The final transfer function, which is in the appendix, is based off the transfer function proposed in Ausubel (2006), but generalized to an arbitrary sequence of prices instead of only an increasing one. An expectation is taken over this transfer function at the end of the mechanism over all possible types who would reach that point in the mechanism in order to form a BIC truth-telling equilibrium.

The other complication in designing a truthful mechanism is that it may terminate before determining an exactly efficient allocation. So, the market need not clear exactly at the final price and therefore some (potentially random) selection rule must exist. In terms of efficiency, this is a relatively straightforward problem to solve: The designer can arbitrarily select between all allocations that maximize revealed utility and obtain the highest efficiency possible. The selection rule can cause more issues in terms of incentives. We will use pre-committed, deterministic tie-breaking to avoid these issues.[9] If ex-ante symmetry is desired, this tie-breaking can be randomized, committed to, and announced at the beginning of the game.

Towards the second point, we first consider the case where $\kappa = L$. Proving that a dynamic price game is c-efficient builds off the fact that when there are gross substitute preferences, any efficient allocation must have supporting clearing prices. Since every efficient allocation must have a supporting price, any terminal steps of a c-efficient mechanism can have a price associated with it. We show that the market clearing in any round τ is always on-path for some type realization θ. We then conclude that soliciting demand in every round is sufficient. The dynamic price game can have any sequence of prices and can condition on past prices arbitrarily and so we can find an offer price and transition function that generates the outcomes of c-efficient, κ-bandwidth mechanism. When $\kappa < L$, we consider only the case where κ is a divisor for L and apply our results for when $\kappa = L$ with the modification that the auction needs the possibility to clear only every L/κ rounds.

Definition 3.1. A **dynamic price game** $((\nu^\tau, s^\tau)_\tau, p^0)$ is defined by the following algorithm:

1. Commit to some priority order over bidders where the higher priority order bidder is always allocated over equivalently allocative efficient bundles.

2. Initialize $\tau = 0$ and $t_i^{-1}, X_i^{-1}, p^{-1} = \mathbf{0}^L$ for all i.

3. Simultaneously ask each bidder i for statistic $s_i^\tau(X_i^\tau; \{X^r\}_{r<\tau})$ of an optimal consumption bundle:
$$X_i^\tau(p^\tau, \theta_i) \in \underset{x_i \in \{0,1\}^L}{\operatorname{argmax}} \{v(x_i, \theta_i) - p^\tau \cdot x_i\}\,.$$

[9] The need for deterministic tie-breaking can be seen in the orderly assumption used in Akbarpour and Li (2020) and Komo et al. (2024) in order to obtain incentive properties.

If $\kappa = L$, let s_i^τ be the identify function.

4. If $\nu^\tau\left((s^r)_{r\leq\tau}\right) = \emptyset$, the game ends and the expected welfare-maximizing allocations is chosen and the transfers are $t_i^\tau((s^r)_{r\leq\tau})$. Otherwise, set

$$p^{\tau+1} = \nu^\tau\left((s^r)_{r\leq\tau}\right),$$

increment τ by 1 and go to step 3.

Theorem 3.2. *There exists a transfer function such that the dynamic price game has a truth-telling equilibrium. Suppose $\kappa \leq L$ and L/κ is an integer. A c-efficient mechanism is allocation equivalent to a truth-telling equilibrium of a dynamic price game.*

Along the way to proving Theorem 3.2, we show a useful facts that when $\kappa = L$, it is without loss to suppose that ν^τ is a function of past aggregate demand. We will build off of Theorem 3.2 by focusing on optimizing within the set of dynamic price games with the truth-telling equilibrium selected for all future results. It is instructive to observe that while we can construct dynamic price games with a XP equilibrium, the c-efficient dynamic price game does not necessarily have one. Considering the case where $L = 1$, observe that in order to have an XP equilibrium, it must be the case that the second highest bidder's value must be observed.[10] However, this information is not necessary to determine the efficient allocation and so it will only be determined in c-efficient dynamic price game if its the most efficient way to determine who has the highest value, i.e., if an ascending auction is the most efficient way to determine the winner then the dynamic price game has an XP equilibrium, but if not then the equilibrium is only BIC.

To assist exposition, for the remainder of the paper we let $\kappa = L$ and $s^\tau = X^\tau$, but the analogous results also hold when $\kappa < L$ and L/κ is an integer.

4 Multiple Equilibria and Activity Rules

In this section, we explore a formal justification for activity rules based on speed and equilibrium refinement. Informally, an activity rule in a dynamic auction is some rule restricting behavior *between* rounds. As a simple example, consider a single-item ascending auction. An activity rule could be that to win item, a bidder must bid (indicate WTP higher than the clock price) in every round, as soon as a bidder "drops out" all her future behavior is ignored.

Perhaps, the most famous activity rule is the Milgrom-Wilson activity rule used in the FCC spectrum auctions. As described in Milgrom (2000), the rule requires bidders to continue to make bids in each round; otherwise their bidding activity is restricted in future rounds. In particular, the auction is a multi-item ascending auction and the activity rule requires that the number of items with bids on them monotonically decreases (with three inactivity waivers allowing for three violations of monotonicity in quantity demanded).[11]

[10] Up to the discretization induced by the cost function.

[11] Formally, for all i and $\tau < \tau'$, $\sum_\ell X_{i,\ell}^\tau \geq \sum_\ell X_{i,\ell}^{\tau'}$.

Another example of an activity rule commonly seen in auctions is early registration requirements at some open outcry auctions. These sorts of rules require participants to arrive and be present for the entirety of the auction process instead of arriving late.

One of the explicit goals of the activity rule in the FCC auction is to "increase the pace of the auction." However, this is somewhat in tension with the formal analysis conducted as this assumed gross substitute preferences and the truth-telling equilibrium. If preferences are gross substitutes and the truth-telling equilibrium is played, then the activity rule will have no effect. In fact, as auctions and activity rule are normally studied, these rules should have no effect because the equilibrium is selected (partial implementation).

To work towards our main result of the section, let us begin by precisely defining a particular type of non-truthful behavior that is plausibly what activity rules are normally meant to prevent.

Definition 4.1. A bidder i **bids lazily** in round τ if she reports $\tilde{X}_i^\tau = X_i^{\tau-1}$ where $X_i^{\tau-1} \neq X_i^\tau$. A **lazy equilibrium** is a equilibrium where there exists i, θ and τ such that i bids lazily in round τ.

We refer to lazy equilibria as such because they represent equilibria where a bidder chooses the status quo instead of updating the message she sends the mechanism. We focus on "lazy" behavior as the focal non-truthfully behavior because it formalizes the idea that bidders are inertial as opposed to having some other reason for non-truthful reporting. Note that by definition of X^{-1}, the only lazy report in period $\tau = 0$ is $\mathbf{0}^L$.

The following theorem shows that any lazy equilibria of c-efficient dynamic price game are strictly less c-efficient than the truth-telling equilibrium. Further, we show that a lazy equilibrium always exists if a c-efficient dynamic price game is XP. XP dynamic price games are important as XP dynamic auctions have been the interest of much of the literature (Nagel and Saitto, 2023; Ausubel, 2006) and influenced the design of, i.e., the FCC spectrum action.

Theorem 4.2. *Consider a c-efficient dynamic price game $(\{\nu^\tau\}_\tau, p^0)$. Then,*

(i) All lazy equilibria are strictly less c-efficient than the truth-telling equilibrium; and

(ii) If $N > L$ and the truth-telling equilibrium is XP, then there exists a lazy equilibrium.

The intuition behind (i) is that a lazy equilibrium has strictly less information communicated about bidders' preferences communicated in some round and so must be strictly less c-efficient, as a c-efficient dynamic price game must be utilizing information as efficiently as possible. In the proof of (ii), we construct a lazy equilibrium by having a bidder who reports lazily in the first round. We enforce this behavior by assuming that off-path all the other bidders will act as if they have types guaranteeing that they demand objects at every possible price queried if the lazy bidder deviates. We take $N > L$ as a technical assumption for our tie-breaking so that we can assign this lazy bidder a low enough priority such that she will not be allocated an item if she deviates. Having the truth-telling equilibrium be XP guarantees that the pay-off from deviating is non-positive.

Observe that this lazy equilibrium does not exist with an activity rule as this punishment scheme is not allowed. Theorem 4.2 shows that there can multiple equilibria in dynamic price games and this multiplicity leads to an "implementation gap" in terms of the designer's objective. This theorem also allows us to see that the intuitive justification of activity rules as speeding up auctions can be formally understood as refining away slow equilibria. In particular, by guaranteeing zero payoff for certain kinds of actions that are not justifiable for truthful agents with gross substitute preferences, this makes such actions strictly dominated and thus makes equilibria requiring these actions non-rationalizable. In this way, Theorem 4.2 reveals a tension between expressiveness and speed: Being expressive allows more non-truthful equilibria to form, all of which are slower than the truth-telling equilibrium. Note that we are not claiming that the rationalizable equilibria under the more lax activity rule are less c-efficient than any allowed under a stricter activity rule, simply that there are more such equilibria. While we only showed (ii) holds for XP equilibria, we conjecture that for any dynamic price game where prices can rise (see Definition 5.1), there exist lazy equilibria. We also do not bound how much less c-efficient non-truthful equilibria are. This means that the non-truthful equilibria could be almost as c-efficient as the truth-telling one. Activity rules allowing only gross substitute preferences could also still result in multiple equilibria.

5 Characterizing The Optimal Mechanism

In this section, we explicitly characterize the c-efficient mechanism. There are two forces to balance for such a mechanism. First, minimizing the worst-case game tree depth. Second, selecting prices that have a high probability of clearing the market immediately. We will first show that with sufficiently large number of bidders and sufficiently thin-tailed value distributions, the second force dominates and a descending auction will be c-efficient. We then discuss how a Bayesian c-efficient mechanism balances these two forces in general.

5.1 Foundations For The Descending Auction

One common, anecdotal reason that people use a descending (Dutch) auction is that it makes goods sell quickly. But, *prima facie*, it is not obvious if a descending auction or an ascending auction (or some other dynamic format) will finish faster. In fact, the descending and ascending auctions have the same worst-case performance. However, the following theorem provides a natural condition under which a descending auction is c-efficient.

Definition 5.1. A dynamic price game $(\{\nu^\tau\}, p^0)$ is a **descending auction** if, for all τ and $(X^r)_{r\leq\tau}$, it is either the case that $\nu^\tau((X^r)_{r\leq\tau}) = \emptyset$ or $\nu^\tau((X^r)_{r\leq\tau}) \leq p^\tau$.

Theorem 5.2. *Suppose that preferences have bound $y^*(\gamma) = o(\log(1/\gamma))$. Then, there exists N^* such that if there are $N > N^*$ bidders, a descending auction will be c-efficient.*

The proof of Theorem 5.2 involves two parts. First, showing that as the number of participants grows, the probability that the market clears at the highest price offered also grows. To prove this, we first show that because preferences are sub-exponential, we can bound the highest queried price as a function of the harmonic series with respect to the number of bidders. Since the harmonic series grows at asymptotic rate $O(\log N)$, having the rate at which the hazard rate increases be exponential means that the probability that the market clears at it's highest value approaches 1. Second, conditional on the market not clearing at the highest price, the probability it clears at the next highest price must also be sufficiently high. This is true because the gap between prices is bounded away from zero and so as the number of bidders increases the probability mass accumulates at its largest mass point.

A few observations are in order about Theorem 5.2. First, the settings in which Theorem 5.2 holds and c-efficiency appears important correspond to settings where descending auctions are common. For example, in commodity auctions, there tend to be a large number of bidders and allocating goods quickly is important because there are many goods to allocate and goods spoil quickly.

Second, this theorem holds when values have unbounded support, as long as preferences are sufficiently thin-tailed. Note that we need a condition that is stronger than just sub-exponential because if preferences are "barely" sub-exponential, the maximum price queried will increase rapidly as N increases. This would cause the probability of the market clearing near its maximum price to approach 0 instead of 1. The bound $y^*(\gamma) = o(\log(1/\gamma))$ intuitively means that the value distribution must have tails about as thin as a Gumbel distribution instead of an exponential distribution. Of course, if the value distribution is bounded, the probability the maximum value is at the highest possible value goes to 1. So, for a sufficiently large number of bidders, the mechanism will begin at a price and descend. If multiple people desire the object at the starting price, there will be inefficiency in allocation. But, the rate of y^* increasing guarantees that for a high enough price, the expected inefficiency will not be very large.

Third, while the results from Wilson (1977) use a similar argument to prove that a posted price is approximately optimal as the number of bidders increases, Theorem 5.2 applies with unbounded support and characterizes the full dynamic mechanism which is c-efficient instead of only an approximation. In fact, our next result will show that we can find instances where the descending auction does at least 33% better than a posted price in terms of c-efficiency.[12]

Proposition 5.3. *Let* $\mathit{Eff}_{PP}(F,N)$ *be the ex-ante expected efficiency of the optimal posted price and let* $\mathit{Eff}_{DA}(F,N)$ *be the corresponding efficiency for the optimal descending auction. Then, there exists* F *and* N *such that a descending auction is ex-ante* c*-efficient and* $\mathit{Eff}_{PP}(F,N)/\mathit{Eff}_{DA}(F,N) \leq \frac{3}{4}$.

Our proof involves providing a bounded support value distribution for a single-item auction and then demonstrating that a descending auction is c-efficient and has the desired bound on $\text{Eff}_{PP}(F,N)/\text{Eff}_{DA}(F,N)$. In particular, even in instances where a posted price will be efficient in the limit, for a finite number of participants, a descending auction can be c-efficient and do better than a posted price. Also note that Proposition 5.3 implies that for some N, Theorem 5.2 holds while a posted price is not a good approximation for the efficient allocation.

[12] The result also holds for allocative efficiency.

5.2 General Case

We now progress to the general case of a c-efficient mechanism. As we have already mentioned, there are two key forces. When the prior is completely uninformative, the maximum game depth should be minimized. When the prior is informative enough, the prior information dominates and the most likely clearing price is queried. For intermediate cases, the designer must trade-off between these two forces.

We can show that, in some sense, the trade-off is one-to-one. The precise formulation of said trade-off can be found in Definition D.1, but we will describe the trade-off informally here. Define the "virtual value" of some price as the probability the dynamic price game terminates after querying that price (the market approximately clears) minus the maximum probability of any residual demand vector. Then, a **divide-and-conquer auction** selects the clearing price that maximizes this virtual value. We prove that being a divide-and-conquer auction is a necessary condition for being Bayesian c-efficient as long as either the type space is infinite or c is sufficiently small. We also discuss in the Appendix how this condition is relatively tight, in the sense that as long as the set of clearing prices from a divide-and-conquer auction can be rationalized under some gross substitute preferences, we can find a prior distribution F such that said divide-and-conquer auction is c-efficient.

There are multiple real-world examples where intuitions from the Bayesian analysis hold. For example, a divide-and-conquer auction will begin with a relatively uninformative prior and so will mostly minimax the space of clearing prices. As the auction progresses, there will be fewer possible clearing prices and more information on demand, and so the prior will become more informative. As the prior becomes more informative, we expect the prices queried to have smaller gaps between them. A real world example of where this phenomenon can be seen is in Sotheby's auctions. The auctioneer will begin at the reserve price and assess demand from the crowd. If demand seems high, the"jumps" between prices will be large. As the auction progresses and demand decreases, the jumps become smaller.

An example of a dynamic auction format that is not purely ascending or descending is the fish auctions used in the Honolulu and Sydney fish markets. In these markets, the starting price is selected to maximize the probability the clearing price is near it and then offered price either ascends or descends depending on demand (Hafalir et al., 2023). As long as the prior over clearing prices is single-peaked, the Honolulu-Sydney auction will be more c-efficient than either the ascending or descending auction. And in fact, the anecdotal evidence suggests that this non-standard auction format is used to try and sell the fish quickly.[13]

6 Complexity Micro-Foundations

In this section we discuss two connections between complexity and speed: Additive costs of considering a large set of actions at once and extending Nagel and Saitto (2023) complexity costs, which are roughly costs from the maximum number of choices ever considered at once, to also have lexographically weaker costs from making a large number of choices.

[13] In the worst-case analysis, the Honolulu-Sydney auction would start at a price such that it is half-way between the maximum plausible clearing price and the reserve price. Empirically, this is not the starting price normally selected.

6.1 Large Menu Costs

In this subsection, we discuss how to microfound our results based on disutility from having to consider large menus. Formally, this means redefining the desideratum in terms of

$$\sum_{i=1}^{N} v_i(z, \theta_i) - \sum_{h \prec z} \tilde{c}(|A(h)|) \tag{2}$$

where A is the set of actions at any particular point in the game, z is an outcome, h is any intermediate state before the outcome, and $\tilde{c}$ is a function representing the distaste for having to consider many actions at once. This is the standard extensive-form game notation; we relegate a complete definition to the Appendix (Definition E.1). The experimental literature supports the notion of menu size disutility: A growing body of lab experiments finds that people make more mistakes (Puri, 2024) and exert more costly effort (Oprea, 2020, 2024) when there are more choices to consider. Notions of complexity of additional actions have also been used to explain real-world phenomenon such as simple demand curves in multi-unit auctions (Kastl, 2011) and short preference lists in school choice.

We will first discuss some general mechanism design results we can obtain using this framing. Then, we will present the additional assumptions needed in order to connect menu costs to price of time. Finally, we give a new application to behavioral economics.

To characterize $\tilde{c}$-efficient mechanisms, the levels of the menu costs are not important, but instead the ratio of the costs of different menu sizes: When costs grow at a super-logarithmic rate, then every $\tilde{c}$-efficient mechanism offers participants a series of binary choices. The formalization is below.

Assumption 6.1. The menu cost $\tilde{c}$ is **super-log**: for all $K > 2$, $\tilde{c}(K) > \lceil \lg K \rceil \cdot \tilde{c}(2)$.[14]

Proposition 6.2. *If a mechanism is $\tilde{c}$-efficient, then bidders are only offered action sets of size two.*

Being super-log is a relatively weak condition on menu costs. For example, requiring that menu costs are super-additive would imply that menu costs are super-log. The choice bracketing literature (Read et al., 2000), supports the assumption that menu costs are super-additive and therefore are super-log.

The proof of Proposition 6.2 comes via contradiction: Given any mechanism where a bidder is offered more than two choices, we can construct a new mechanism where bidders are offered a series of binary choices that is strictly more $\tilde{c}$-efficient. Note that the converse of Proposition 6.2 is not true. If certain outcomes are far more likely than others, offering a single menu may not be optimal when costs are sub-log.[15]

To understand the relationship between $\tilde{c}$-efficiency and fast price discovery we require only one additional assumption: that all participants are presented their choices simultaneously. This corresponds to the assumption in our baseline model that there is zero marginal cost of asking an additional participant for information. More formally,

[14] $\lg K \equiv \log_2 K$.

[15] This difference is conceptually related to the fact from the theoretical computer science literature that two-way communication protocols can be much more efficient than one-way communication protocols (Roughgarden et al., 2020).

Assumption 6.3. We only consider mechanisms that satisfy **simultaneity**: whenever one participant takes an action, all participants take an action.

Once we make this assumption, there is now a correspondence between the $\tilde{c}$-efficiency and dynamic price games, analogous to Theorem 3.2. In particular, all $\tilde{c}$-efficient mechanisms are allocation equivalent to a dynamic price game with the further restriction on the transition rules that they can only use N bits of information from each aggregate demand to determine the next price:

Corollary 6.4. *If a mechanism is $\tilde{c}$-efficient, then there exists a dynamic price game $\left((\nu^\tau, s^\tau)_\tau, p^0\right)$ that implements the same allocations.*

Note that this is true for both Bayesian and worst-case notions of efficiency. The proof builds off Proposition 6.2 to argue that every "round" each bidder gives a single bit of information. Thus, the $\tilde{c}$ induces a 1-bandwidth mechanism and we can apply Theorem 3.2 to complete the proof. Thus, the fact that dynamic price mechanism are so common is robustly explained by our theory. We can also derive the following two corollaries of Corollary 6.4:

Corollary 6.5. *Suppose $L = 1$. An implementable outcome is $\tilde{c}$-efficient if and only if it is c-efficient for $c(K) = KN\tilde{c}(2)$.*

Corollary 6.6. *Suppose that preferences have bound $y^*(\gamma) = o(\log(1/\gamma))$. Then, there exists N^* such that if there are $N > N^*$ bidders, a Bayesian $\tilde{c}$-efficient mechanism is allocation equivalent to a descending auction.*

Corollary 6.5 shows that in the single-item case, menu costs (that are super-log) are the same desideratum as speed considerations. Corollary 6.6 shows our primary result from Section 5, that descending auctions are best when there are sufficiently large numbers of participants, also holds when considering efficiency in terms of complexity costs instead of speed.

6.2 Nagel and Saitto (2023) Complexity

Nagel and Saitto (2023) define complexity in terms of the maximum number of different choices a bidder ever has to face at one time (ignoring choices that immediately guarantee her a specific outcome, which they refer to as *clinching actions*). They restrict to only considering DS equilibra and discrete, finite type distributions. For example, while the single-item Vickrey auction and the English auction both have a DS equilibrium, the Vickrey auction has a complexity score equal to the number of types $|\Theta|$ and the English auction has a complexity score of 2. This is because in the single action bidders take in the Vickrey auction, they must announce a type, whereas in the English auction they merely have to announce if their type is above some threshold value. When $L = 1$, all dynamic price games have minimum complexity (complexity of 2). For homogeneous goods, they show that the Ausubel (2004) auction has minimum complexity (complexity of L).

For single-item auctions, if we assume finite types and take $\sup c(K) \to 0$, the results of Nagel and Saitto (2023) hold in our setting. In other words, if the designer first minimizes Nagel and Saitto (2023) complexity and then breaks ties in terms of the number of actions that must be taken, then our results still hold. We should highlight a few distinction when comparing the notion of complexity here to that of Nagel and Saitto (2023). First, they restrict to DS equilibria, while we allow for any BIC equilibrium. This allows our characterization of speed as a desideratum to be over a larger set of problems including allowing us to make progress on the heterogeneous item case. Second, we generalize the value space to allow for infinite type spaces. Third, the class of mechanisms selected here is much smaller than that in Nagel and Saitto (2023), i.e., our desideratum is stricter.[16]

7 Discussion

7.1 Willingness-to-Pay Elicitation

A behavioral application that is similar to a price discovery process is willingness-to-pay (WTP) elicitation. WTP elicitation is important in many lab settings in order to determine, for example, the value of an intervention or certainty equivalent for a lottery. This elicitation has the flavor of price discovery in the sense that in a c-efficient mechanism, we want to determine a rough estimate of the highest value for the good. Here, we want to obtain a rough estimate of the single agent's value for the good. We may think that speed is not an important factor, but Corollary 6.4 shows how we can use what we learned about κ-bandwidth mechanisms to design mechanisms focused on menu complexity.

Consider the Becker–DeGroot–Marschak (BDM) method of eliciting WTP. A BDM process simulates a second-price auction: The participant is asked what her WTP is (in some way), and then that value is compared to some randomly drawn value b. If her WTP is higher than b, she gets the item. Otherwise, she gets paid b. There are two common variants of BDM methods: single-menu BDM where a single menu of different WTPs are presented and the participant selects one; and binary-choice BDM where the participant is presented a series of binary choices asking if her value is above some threshold.[17]

Observe that for both single-menu BDM and binary-choice BDM, truth-telling is a dominant strategy. Our theory provides guidance on when to choose different variants. More formally, let $\theta \in [\underline{\theta}, \overline{\theta}]$ to be the value for the good in the WTP elicitation and have $v(z,\theta) = x(z) \cdot \theta + t(z)$. We can apply Proposition 6.2 to conclude that when the costs are super-log, the $\tilde{c}$-efficient mechanism is a binary-choice BDM (for any prior).

7.2 Worst-Case Analysis

For most of this paper, we have considered a designer with a prior over values. In this subsection, we analyze the structure of c-efficient price discovery mechanisms when the designer is instead worried about the worst-case scenario. For ease of exposition, we will assume that the type space is finite. Formally, in this subsection we consider the following desideratum:

[16] We also do not distinguish clinching actions from other sorts of actions, but that is not important for these results.

[17] When all choices are presented on the same screen, binary choice BDM is known as a multiple price list.

Definition 7.1. A mechanism (x, t, σ) is **worst-case c-efficient** if

$$(x, t, \sigma) \in \underset{\substack{x':\mathcal{M}\to\Delta^N(\{0,1\}^L),\\ t':\mathcal{M}\to\mathbb{R}^N,\\ \sigma'\in\Sigma_{BIC}}}{\text{argmax}} \left(\min_{\theta\in\Theta^N} \left[\sum_{i=1}^{N} v(x_i'(\sigma'(\theta)), \theta) - c\left(K^*(x', t', \sigma', \theta)\right) \right] \right).$$

We will characterize an asymptotically worst-case c-efficient mechanism in this subsection and leave the full characterization to the appendix. Let us begin describing the c-efficient mechanism for the single-item case to help build intuition. When the item is exactly demanded at any price, the market immediately clears in any c-efficient mechanism. Thus, the designer can minimize the maximum depth focusing only on the item being over and under demanded. This immediately yields the optimal solution: a binary search tree. The c-efficient mechanism begins by offering the price that splits the support of values in half. Then, if the market is over-demanded (at least two people want the good), the next price bifurcates the support in the top half. If under-demanded, the bottom half is bifurcated instead. This process is repeated until the market clears or the size of the support is sufficiently small (less than $\underline{c}$ width).

Moving on to the multi-item case, it is helpful to think of the offered price as a vector in an L-dimensional space. Then, the demand in the market at that price generates another L-dimensional vector which is $+1$ in dimension ℓ if item ℓ is over-demand, -1 if the item is under-demanded, and 0 otherwise. This vector restricts what the possible clearing prices are. For example, if every good is over-demanded then the clearing price must be strictly higher in every dimension than the current price. We now define our mechanism of interest: the simultaneous binary auction. This auction performs a binary search in each dimension independently. To formalize our result, let $d^*(p, \theta) = sgn\left(\overline{X}^*(p, \theta) - \mathbf{1}^L\right)$ to be the directional vector of residual demand if the realized type profile is θ and the price is p and let the set of possible exact clearing prices be

$$P^\tau\left(p, d, \{\overline{X}^r\}\right) = \left\{p^*(\theta) : \forall r \leq \tau : \overline{X}^*(p^r, \theta) = \overline{X}^r \text{ and } d^*(p, \theta) = d\right\}.$$

Then let the set of prices that could clear item ℓ above some given price p_ℓ be

$$\overline{P}_\ell^\tau(p_\ell) = \left\{ \tilde{p}_\ell > p_\ell : \tilde{p} \in \bigcup_{d\in\{-1,0,1\}^L} P^\tau\left(p^\tau, d, \{\overline{X}^r\}\right) \right\},$$

and the set below be

$$\underline{P}_\ell^\tau(p_\ell) = \left\{ \tilde{p}_\ell < p_\ell : \tilde{p} \in \bigcup_{d\in\{-1,0,1\}^L} P^\tau\left(p^\tau, d, \{\overline{X}^r\}\right) \right\}.$$

The formal definition is then,

Definition 7.2. A dynamic price game $(\{\nu^\tau\}, p^0)$ is a **simultaneous binary auction** if, for all τ and $\{\overline{X}^r\}_{r=1,\ldots,\tau}$, it is either the case that $\nu^\tau(\{\overline{X}^r\}) = \emptyset$ or for all ℓ,

$$\nu^\tau\left(\{\overline{X}^r\}\right)_\ell \in \begin{cases} \text{argmin}_{\tilde{p}_\ell} \left|\left|\overline{P}^\tau_\ell(\tilde{p}_\ell)\right| - \left|\overline{P}^\tau_\ell(p^\tau_\ell)\right|/2\right| & \overline{X}^\tau > 1 \\ \{p^\tau_\ell\} & \overline{X}^\tau = 1 \\ \text{argmin}_{\tilde{p}_\ell} \left|\left|\underline{P}^\tau_\ell(\tilde{p}_\ell)\right| - \left|\underline{P}^\tau_\ell(p^\tau_\ell)\right|/2\right| & \overline{X}^\tau = 0 \end{cases}. \tag{3}$$

When preferences are additive, this auction exactly coincides with the c-efficient auction, but for other possible preferences, the simultaneous binary auction is less efficient for two primary reasons. First, if preferences are not additive, demand for one good can restrict the possible demand for another good. Second, when $d_\ell = 0$, the set of possible prices for item ℓ is not just the current p_ℓ and so the c-efficient auction should try to split the search space for item ℓ, which the simultaneous binary auction does not. Nevertheless, the following result shows that the simultaneous binary auction has the same asymptotic performance as the c-efficient auction and both finish much faster than the descending auction does in the worst-case.

Proposition 7.3. *Suppose there are M possible clearing prices per item. The worst-case number of steps for both the simultaneous binary auction and c-efficient mechanism is $O(\log M)$. The worst-case number of steps for the descending auction is $O(L \cdot M)$. There exist a type space Θ such that these asymptotic bounds are also lower bounds.*

We explicitly characterize the c-efficient mechanism in Definition F.1. We refer to these auctions as **L-nary auctions** and instead of computing prices for each good separately, price vectors are selected such that the maximum L_1 distance between possible clearing prices is minimized in each step.

The key step in the proof of Proposition 7.3 involves demonstrating that we can find p such that for all d, the size of $P^\tau(p, d, \cdot)$ is at most a 2^{-L} fraction of the size of all possible clearing prices at step τ. We can then find gross substitute preferences such that the asymptotic number of queries from Proposition 7.3 are lower bounds and so the upper bounds found are tight. This worst-case is natural: It is additive preferences and the types are realized such that the maximum price clears the market. Additive preferences induce this performance because demand for one good provides no constraints on demand for another good and so independent binary search, which is $O(\log M)$, is the best that can be done.

Observe that the asymptotic runtime for the simultaneous binary auction is only a function of the number of possible clearing prices per item, i.e, as the number of items increases, the asymptotic runtime of the simultaneous binary auction is constant. This is because as the number of items increases, the amount of information that can be transmitted in each step also increases. This result complements Theorem 5.2 as it suggests that, especially if the number of items is large compared to the number of bidders, changing from a descending auction to a different auction format can have dramatic speed-ups. Note that ascending auctions have the same asymptotic worst-case runtime as descending auctions.

To conclude this subsection, we provide an example of the implementation of a simultaneous binary auction.

Example 7.4. Consider a two item ($L = 2$) allocation problem. Suppose that Θ is such that bidders are single-minded: For all $\theta \in \Theta$, it is the case that $v_1(\theta) \equiv v(\{1\}, \theta) \in \{1, \ldots, 8\}, v_2(\theta) \equiv v(\{2\}, \theta) \in \{1, \ldots, 8\}$ and $v_{12}(\theta) \equiv v(\{1, 2\}, \theta) = \max\{v_1, v_2\}$. Suppose that $c = \frac{1}{10}$ so that all possible prices are distinguished in a c-efficient mechanism. For an ascending auction, the worst-case clearing price is $(8, 8)$. In this case, we must ascend prices one item at a time and so the total time cost is $\frac{7}{5}$. For the simultaneous binary auction, a worst-case clearing price is again $(8, 8)$. A sequence of prices to clear the market is $(4, 4), (6, 4), (7, 6), (8, 7), (8, 8)$ for a total time cost of $\frac{1}{2}$. An efficient L-nary auction only does slightly better: $(4, 4), (7, 5), (8, 7), (8, 8)$ for a total time cost of $\frac{2}{5}$.

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A Section 2 Appendix

Proof of Lemma 2.8

To complete this proof, we will show that $\overline{K}(y^*)$ is uniformly bounded for all type distributions F that induce sub-exponential preferences. This will mean that for any ex-ante or worst-case c-efficient mechanism, there is a finite bound on the number of steps in the mechanism. First, let us prove that for all x, there exists $\check{y}_x$ such that for all $y > \check{y}_x$,

$$\mathbb{E}_{\theta_i \sim F}\left[\max_{i \in \{1,\dots,N\}} v(x,\theta_i) \mid \max_{i \in \{1,\dots,N\}} v(x,\theta_i) \geq y\right] - y < \frac{\underline{c}}{2L}. \tag{4}$$

To see this is true, first observe that if there exists y^* such that $F_x(y^*) = 1$, Equation (4) is trivially satisfied for $\check{y}_x = y^*$ as the LHS of Equation (4) will always be 0. When instead the hazard rate is strictly bounded beyond some $y^*(1/\gamma)$, we can use comparisons to the exponential distribution to prove results. Recall that the expected max of N independent draws of an exponential distribution with rate λ is $\frac{1}{\lambda}\sum_{i=1}^{N}\frac{1}{i} \leq \frac{N}{\lambda}$. So,

$$\mathbb{E}_{z_i \sim \exp(\lambda)}\left[\max_{i \in \{1,\dots,N\}} z_i \mid \max_{i \in \{1,\dots,N\}} z_i \geq z^*\right] - z^* < \frac{N}{\lambda}. \tag{5}$$

Then, since F_x is sub-exponential, we can set $\check{y}_x = y^*\left(\frac{2NL}{\underline{c}}\right)$ and then use Equation (5) to conclude that $\check{y}_x$ satisfies Equation (4). Let $\check{y} = \max_{x \in \{0,1\}^L}\{\check{y}_x\}$.

Suppose that our mechanism cannot distinguish any values above $\check{y}$, simply treats them all as the same value and allocates arbitrarily between bundles with this assumption. Applying Equation (4), the expected efficiency loss per bundle allocated is at most $\frac{\underline{c}}{2L}$ and since the most number of bundles that could be allocated is L (each item goes to a different person), the total efficiency loss is at most $\underline{c}/2$. In order to distinguish values above $\check{y}_x$ would incur a time cost of at least $\underline{c}$ since at least one additional query must be made. This is clearly sub-optimal in expectation. By a similar argument, values for bundles that are within $\frac{\underline{c}}{2L}$ will never be distinguished by a c-efficient mechanism. So each bundle has at most $\check{y} \cdot \frac{2L}{\underline{c}}$ possible prices. Thus, defining

$$\overline{K}(y^*) = \check{y} \cdot \frac{2L}{\underline{c}} \cdot 2^L,$$

we can conclude that $\overline{K}$ is a bound on the number of steps on the number of steps for any c-efficient mechanism. This is by appealing to Theorem 3.2 and observing that there are at most $\overline{K}(y^*)$ possible prices. □

B Section 3 Appendix

Proof of Theorem 3.2

We prove this theorem by first constructing a transfer rule such that truth-telling in the dynamic price game is BIC. Then, we prove restricting to dynamic price games is without loss to restrict with respect to c-efficiency.

Constructing the Equilibrium. To begin constructing the equilibrium, let $x_i^*(\theta)$ be the realized allocation vector for bidder i when the type realization is θ. Then, let us define the following transfer rule. Let $R_i^\tau = sgn(-\mathbf{1}^L + \sum_{j \neq i} X_j^\tau)$ be the sign[18] of the residual demand for an item excluding bidder i. Define ex-interim transfers as

$$t_i^\tau = t_i^{\tau-1} + \sum_{\ell=1}^{L} x_{i,\ell}^* \cdot p_\ell^\tau \cdot \left[R_{i,\ell}^{\tau-1} - R_{i,\ell}^\tau\right].$$

Then, consider a modified price transition rule $\hat{\nu}^\tau$ where if there exists $\overline{X}$ such that $\nu^\tau(\overline{X}) \neq \emptyset$, then $\hat{\nu}^\tau \neq \emptyset$. Instead, every possible future clearing price is queried for demand. Then, the final transfers in the original game ν^τ are

$$t_i^*(\theta) = \mathbb{E}\left[\hat{t}_i \mid (p^r, X^r)_{r \leq \tau}\right]$$

where $\hat{t}$ is defined analogously to t but for $\hat{\nu}$. We now show that $\hat{\nu}$ has an XP truth-telling strategy. This is sufficient for proving that the original dynamic price game has an BIC truth-telling strategy, since the allocations are equivalent in the two games and we are taking the appropriate expectation to compute t from an XP equilibrium.

To prove that $\hat{\nu}$ has an XP truth-telling equilibrium, consider bidder i and fix the other bidders to tell the truth. Suppose bidder i has type θ_i and deviates to playing as if she has type θ_i'. First observe that by construction, if the allocation does not change, then the transfers do not change in $\hat{\nu}$. Since we are considering an XP equilibrium, consider only type realizations such that this deviation causes a change in the allocation. Observe that since all prices are queried, the mechanism observes the demand vectors from all bidders at all possible clearing prices. So, by definition, the change in transfers paid is the change in the allocation multiplied by the change in the price other bidders are willing to pay for that change. However, by the assumption that other bidders are truthfully reporting and that the mechanism is allocating as efficiently as possible, the change in what others are willing to pay must be less than a bidder with type θ_i is willing to pay for that additional bundle under the proposed allocation when she deviates to play like θ_i'. Likewise, if her allocation increases after deviating, it must be the case that other bidders value the additional allocation more than i. Observe that we need the consistent tie-breaking to rule out an edge case: a deviation could affect which tie-breaking rule is used among bundles that the mechanism are equally efficient in expectation but differ in value from i's perspective.

Restricting to Dynamic Price Games. Begin by considering $\kappa = L$. Towards contradiction, suppose there exists a c-efficient mechanism $\mathcal{M}$ that achieves higher c-efficiency then any dynamic price game. Recall that we only need to focus on allocations and not transfers when considering c-efficiency. It is without loss with respect to allocations to restrict to

[18]We define the sign function $sgn : \mathbb{R}^L \to \{-1, 0, 1\}^L$ dimension-wise: for all $\ell \in \{1, \ldots, L\}$,

$$sgn(x)_\ell = \begin{cases} 1 & x_\ell > 0 \\ 0 & x_\ell = 0 \\ -1 & x_\ell < 0 \end{cases}.$$

ex-ante as opposed to ex-post (or ex-interim) randomization. This is because any ex-post randomization over allocations can be implemented by ex-ante randomizing over a vector where each element of the vector is a different ex-post randomization. This randomization achieves the same c-efficient as ex-post randomizing, although we might have to relax incentive compatibility constraints to achieve it. So, we can just pick an arbitrary draw from the ex-ante randomization and suppose that $\mathcal{M}$ is deterministic. If we can demonstrate that a dynamic price game is weakly more c-efficient than deterministic $\mathcal{M}$ without considering the incentive compatibility of $\mathcal{M}$, then we will have completed this section of the proof.

The next step of the proof is to show that each termination step of $\mathcal{M}$ can be associated with an approximate clearing price. If $\mathcal{M}$ is c-efficient, then the allocation must maximize allocative efficiency given current information on participants' values. Observe that there must be some price (with some tie-breaking rule) that gives the same allocation. Towards contradiction, suppose this was not true. If the allocation is efficient, we apply our assumption that preferences satisfy gross substitutes to claim there exists a clearing price p^* that implements the same allocation (Nisan and Segal, 2006). So if there did not exist some price that gave the same allocation as $\mathcal{M}$, then the market does not clear exactly. But, that means for every price and tie-breaking rule that could be offered, the allocation is not achievable. Thus, if the market clears exactly, there must be an associated clearing price. Since the allocation cannot directly condition on type, only on bidders' actions, if the market can exactly clear at the end of some round τ, we can associate that round with the appropriate clearing price and truthful reporting of preferences at that bundle are a sufficient statistic to determine if the proposed price clears the market and thus we can obtain the same allocation in round τ of a dynamic price game and $\mathcal{M}$.

Next, we consider approximating efficient allocations. Recall that we are focused on pure strategies and so after each round, the actions that participants take rule out a portion of Θ with probability 1. Let $\widetilde{\Theta}$ be the possible types at the termination point. By definition, any c-efficient mechanism must select an allocation in the set that achieves the highest allocative efficiency given the current information on possible types $\widetilde{\Theta}$:

$$\underset{x}{\operatorname{argmax}}\, \mathbb{E}_{\theta\sim F|\theta\in\widetilde{\Theta}}\left[\sum_{i=1}^{N} v_i(x_i,\theta_i)\right].$$

By definition, every dynamic price game does this. In this case, we can associate any price we would like with the final allocation.

The next step is to prove that it must be possible for the market to clear at every step and therefore that we can associate every node with a clearing price. Towards contradiction, suppose there exists some type realization θ^* and round τ^* such that no matter what actions bidders take in $\mathcal{M}$, the mechanism will not end. Let $\{p^1, p^2, \ldots, p^K\}$ be the set of possible clearing prices. Observe that each price p^k is associated with $\Theta_k = \left\{\theta : p^*(\theta) = p^k\right\}$. If a mechanism is c-efficientand can never clear at τ^*, it means that for any message sent m (of the $2^L \times N$ possible messages), letting $\widetilde{\Theta}_m$ be set of possible messages after that, there exists $k \neq k'$ such that

$$\widetilde{\Theta}_m \cap \Theta_k \cap \Theta_{k'} \neq \emptyset.$$

Since the message space is large enough, there must exist an allocation, and therefore k, such that we could construct a message m^* such that $\widetilde{\Theta}_{m^*} \subseteq \Theta_k$. We keep the information sets from the other messages $\widetilde{\Theta}_{-m^*}$ the same as in the original mechanism. This mechanism is clearly more c-efficient since it clears faster for every type in Θ_k and equally as fast for all other types. But, $\mathcal{M}$ is c-efficient and thus we have derived a contradiction.

The final step is to show that the sequence of demands and the initial price are sufficient statistics to generate future prices for outcomes in $\mathcal{M}$. This is straightforward: from the initial price p^0 and initial demand X^0, there would only be one price offered next in $\mathcal{M}$ necessarily since $\mathcal{M}$ is deterministic. So, we can derive all prices to τ from the sequence of demand. This means we can recover all information contained in the mechanism. Note that since we only care about allocative efficiency, participants are symmetric, and truth-telling is an equilibrium, we only need to track aggregate demand and not the entire demand matrix.

Thus, we have constructed a dynamic price game. We have already demonstrated that there exists a truth-telling equilibrium, meaning that $(\{\nu^\tau\}, p^0)$ implement the same outcomes as $\mathcal{M}$, deriving a contradiction and completing the proof for when $\kappa = L$.

Now consider $\kappa < L$. By the same logic as above, the market must be able to clear at least every L/κ rounds. To complete the proof, we just have to prove that statistics of demand are sufficient statistics for any c-efficient mechanism. Between any two rounds that the mechanism could stop and efficiently allocate the goods, demand at some price must be completely distinguished. If this is not the case, then the market could not clear efficiently with ex-post probability of 1 as there must be residual uncertainty of the efficient allocation between rounds otherwise a c-efficient mechanism must terminate. Therefore, the information accumulated between possible clearing rounds must be able to be accumulated into a demand at a clearing price, and given the frequency in which the market must clear, only information about demand can be communicated and so demand is a sufficient statistic. □

C Section 4 Appendix

Proof of Theorem 4.2

Proof of (i). Consider any lazy equilibrium. We first show that for price path, the truth-telling equilibrium is at least as c-efficient as this lazy equilibrium in expectation. We then show that there must exist a price path where the inequality is strict.

Towards contradiction, suppose there exists a price path where the lazy equilibrium is more c-efficient in expectation. Then, the dynamic price game cannot be c-efficient: construct a new dynamic price game $(\{\tilde{\nu}^\tau\}_\tau, p^0)$ that solicit inputs truthfully but maps reports in $\tilde{\nu}$ to the outcomes in the lazy equilibrium of the original game. Truth-telling is an equilibrium because outcomes are the same as the lazy equilibrium in the original game. Thus, $(\{\tilde{\nu}^\tau\}_\tau, p^0)$ is more c-efficient than $(\{\nu^\tau\}_\tau, p^0)$ which implies $(\{\nu^\tau\}_\tau, p^0)$ is not c-efficient.

To show the inequality is strict, observe that since every price path has the truth-telling equilibrium weakly dominate the lazy one, in order for the equilibria to be equally as c-efficient, then both equilibria must yield the same outcomes. However, this either implies that (1) the game is not c-efficientor (2) a bidder's lazy bidding is never relevant to the outcome. Consider case 1. If the lazy bidding in round τ does not change the outcome and differs from the truthful report, then that implies that if the game ends in τ the outcome is the same under both reports. But this means that the game is not c-efficient because the aggregate demand for at least one item must change and since the equilibrium play by bidder i cannot condition on bidder $j \neq i$, this must change the efficient outcome for positive measure of types. In order for lazy bidding to never be pivotal compared to truthful bidding, that would imply that truthful and lazy bidding yield the same efficient allocation. But this would imply that truthful bidding has to be equivalent to lazy bidding, which contradicts our definition of lazy bidding having to be different than truthful bidding.

Proof of (ii). We begin by proposing a lazy equilibrium and then show that it always exists. Let $N > L$ and without loss assume that the priority order is lexicographic. Define bidders 1 through $N-1$ as the active bidder and bidder N as the lazy bidder. In the first round, the lazy bidder N reports lazily ($X_i^0 = \mathbf{0}$. In all other rounds, she reports truthfully. Each active bidder i reports truthfully in the first round. If bidder N reports lazily, $X_N^0 = \mathbf{0}$, then all active bidders report truthfully for the rest of the game. Otherwise, each active bidder "bids up" all items with residual demand and reports $X_i^\tau = \mathbf{1}$ for the rest of the game.

To prove this is an equilibrium, first consider the lazy bidder. Since this bidder N has the lowest priority order, if the market does not clear immediately and the game continues beyond a single round, she will receive no allocation in equilibrium. So, the only potentially profitable deviations come from considering reports that will clear the market immediately. In order for such a deviation to be profitable, it must be the case that the value of winning items now times the probability of winning said items is higher than the expected value of winning them later (conditional on the market not clearing). Observe that if a bundle would have been under-demanded if bidder N reported lazily in the first round, then in expectation—assuming truthful reporting—the transfers will be less in future rounds. Recall that dynamic price games conditions transfers assuming truthful reporting and so actual expected transfers for a bundle will be less in future rounds. So, to complete the proof that this deviation is not profitable, we have to show that the probability of winning the bundle in future rounds is weakly higher than in the first round. Do this, recall that we are only considering dynamic price games where truthful reporting is an XP equilibrium and observe that if this were not the case then it would be better to misreport demand at p^0 for the bundle of interest for some realizations of θ_{-N}. In future rounds, truthful reporting is also an equilibrium strategy. This is because truthful reporting is an equilibrium strategy no matter the type realization by assumption of XP equilibrium. Further, the final allocation and transfers must correspond to some type vector as we have assumed the truthful equilibrium is c-efficientand having outcomes that do not correspond to some type would be inefficient. Thus, truthful reporting will be XP for each bidder.

To prove that truthful reporting is an equilibrium by the active bidders, recall that in the first round, truthful reporting must be an equilibrium strategy because the lazy bidder plays as if she has type $\theta_N = \underline{\theta}$ and truthful reporting is XP. As above, truthful reporting in other rounds must be an equilibrium strategy as truthful reporting is an XP equilibrium, no matter the price sequence.

□

D Section 5 Appendix

Proof of Theorem 5.2

To begin this proof let us consider the case where the preference distribution is unbounded. We will return to the bounded distribution at the end of the proof. We first consider the rate at which the expectation maximum increases as a function of the number of participants. For an exponential distribution with rate λ, the expectation of the maximum of N i.i.d. draws is

$$\mathbb{E}_{\theta_i \sim \exp(\lambda)}\left[\max_{i=1,\ldots,N} \theta_i\right] = \frac{1}{\lambda}\sum_{i=1}^{N}\frac{1}{i}.$$

Since exponential distributions are memoryless, we therefore know that for any y, the following is true:

$$\mathbb{E}_{\theta_i \sim \exp(\lambda)}\left[\max_{i=1,\ldots,N} \theta_i \mid \max_{i=1,\ldots,N} \theta_i \geq y\right] - y \leq \frac{1}{\lambda}\sum_{i=1}^{N}\frac{1}{i}.$$

Let us define $\gamma_N = \frac{\underline{c}}{LH_N}$ where $H_N = \sum_{i=1}^{N}\frac{1}{N}$ is the N-th harmonic number and observe that

$$\mathbb{E}_{\theta_i \sim \exp(1/\gamma_N)}\left[\max_{i=1,\ldots,N} \theta_i \mid \max_{i=1,\ldots,N} \theta_i \geq y\right] - y \leq \frac{\underline{c}}{LH_N}\sum_{i=1}^{N}\frac{1}{i} = \frac{\underline{c}}{L}.$$

Thus, for any sub-exponential preferences with bound y^*, if $y > y^*(\gamma_N)$, we can bound the maximum conditional on being above y with N participants. In particular, we will never distinguish values above $y^*(\gamma_N)$. Note that $\gamma_N \to 0$ as $N \to \infty$ because $H_N \to \infty$. So, by our assumption that $y^* = o(\log(1/\gamma))$, this means that for all $A > 0, \alpha > 0$, when N is large enough $y^*(\gamma_N) \leq -A\log(\alpha \cdot \gamma_N)$. It also means that $h(t) \geq \frac{1}{\alpha}\exp(t/A)$ for sufficiently large t. Thus,

$$\begin{aligned}
\mathbb{P}\left[\max_{i=1,\ldots,N+1} v(x,\theta_i) < y^*(\gamma_N)\right] &= \left(F_x(y^*(\gamma_N))\right)^N \\
&= \left(1 - \exp\left(-\int_0^{y^*(\gamma_N)} h(t)dt\right)\right)^N \\
&\leq \left(1 - \exp\left(-\frac{L}{\underline{c}A}H_N\right)\right)^N \\
&\leq \left(1 - \exp\left(-\delta H_N\right)\right)^N \text{ for some } \delta < 1.
\end{aligned}$$

Recall that

$$\lim_{N\to\infty}\left(1 - \exp\left(-H_N\right)\right)^N = \exp(-1/\gamma_{EM}) \approx 0.18$$

where γ_{EM} is the Euler–Mascheroni constant. Thus,

$$\lim_{N\to\infty}\left(1 - \exp\left(-\delta H_N\right)\right)^N < \frac{1}{2} \text{ when } \delta \text{ is sufficiently small.}$$

We can take δ to be arbitrarily small by taking A to be sufficiently large (and therefore N large). We can therefore conclude that when N is sufficiently large, with probability greater than $\frac{1}{2}$, the clearing price in the market will be $y^*(\gamma_N)$ for each item. Thus, a c-efficient mechanism must try to clear the market at this price first. Conditional on the market not clearing at $y^*(\gamma_N)$, that probability that the maximum valuation is far from $y^*(\gamma_N)$ is

$$\mathbb{P}\left[\max_{i\in\{1,\dots,N+1\}} v(x,\theta_i) < y^*(\gamma_N) - \underline{c}/L\right] = \left(1 - \exp\left(-\frac{L}{\underline{c}A} H_N \exp(-\underline{c}/L)\right)\right)^N.$$

Observe that $\exp(-c/L) < 1$, and so for A sufficiently large, it is the case that

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{i\in\{1,\dots,N+1\}} v(x,\theta_i) < y^*(\gamma_N) - \underline{c}/L \mid \max_{i=1,\dots,N+1} v(x,\theta_i) < y^*(\gamma_N)\right] < \frac{1}{2}. \tag{6}$$

Since the preferences satisfy gross substitutes, the efficient allocation will be allocating any given participant at most one good with arbitrarily high probability as $N \to \infty$. Therefore Equation (6) implies that if the market does not clear, we simply lower the price for all goods that were under-demanded by at least $\underline{c}/L$. We can repeat this argument to show that we only want to decrease the prices of goods that are under-demanded and when N is sufficiently large this will not change demand for other goods and therefore the prices will only descend in a c-efficient mechanism.

Note that when the distribution is bounded the same proof holds by observing that as $N \to \infty$, the probability that the market clears at its highest price must go to 1. This is because $a^N \to 0$ for $a < 1$. We can then observe that conditional on an item not clearing at its maximum price, the probability that it clears at the next highest price must go to 1. This is by the same argument. as there will be positive probability it clears at the next highest price; in the discrete case because the next highest price will have positive probability, in the continuous case because there will be an at least $\underline{c}/L$ gap between the two prices and there will be positive probability of the market clearing in those prices. We then can exactly follow the argument from the unbounded case to conclude that a descending auction is c-efficient in this case. $\square$

Proof of Proposition 5.3

Consider a single-item auction. Let F induce three atoms in value space: 0, v_1, and v_2 with probabilities p_0, p_1, and p_2, respectively. Now, suppose v_1 and v_2 are large enough that a c-efficient descending auction distinguishes all values. Let $p_{N,k} = \mathbb{P}\left[\max_{i\in\{1,\dots,N\}} v_i = v_k\right]$. In particular, $p_{N,2} = 1-(1-p_2)^N$ and $p_{N,1} = (1-p_2)^N - p_0^N$. The ex-ante expected c-efficiency of the descending auction is lower bounded by

$$\mathrm{Eff}_{DA}(F,N) \geq p_{N,2}v_2 + p_{N,1}v_1 - 2\underline{c}. \tag{7}$$

If the posted price p is such that $p \in (v_1, v_2]$, then the expected welfare is

$$\mathrm{Eff}^H_{PP}(F,N) = p_{N,2}v_2 + (1-p_{N,2})\frac{p_1}{1-p_2}v_1 - \underline{c}, \tag{8}$$

and if $p \in (0, v_1]$, the welfare is

$$\mathrm{Eff}^L_{PP}(F,N) = (1-p_{N,0})\left[\frac{p_1}{1-p_0}v_1 + \frac{p_2}{1-p_0}v_2\right] - \underline{c}. \tag{9}$$

Combining Equations (7) to (9), we see that

$$\frac{\mathrm{Eff}_{PP}(F,N)}{\mathrm{Eff}_{DA}(F,N)} \leq \frac{\max\left\{\mathrm{Eff}^{L}_{PP}(F,N), \mathrm{Eff}^{H}_{PP}(F,N)\right\}}{\mathrm{Eff}_{DA}(F,N)} \tag{10}$$

Now, let $N^* = 400$ and F^* such that the atoms are $\{0, 0.0411v, v\}$ with probabilities $p_0 = 0.919$, $p_1 = 0.08$, and $p_2 = 0.001$. Some computation will reveal that $\mathrm{Eff}^{H}_{PP}(F^*, N^*) > \mathrm{Eff}^{L}_{PP}(F^*, N^*)$ and then taking $v = 100c$, we can verify from Equation (10) that

$$\frac{\mathrm{Eff}_{PP}(F^*,N^*)}{\mathrm{Eff}_{DA}(F^*,N^*)} \leq \frac{3}{4}.$$

Further, because $p_{N,2} > 1/2$ and $p_{N,1} > p_{N,0}$, the c-efficient auction is clearly a descending auction. □

D.1 General Bayesian Analysis

In order to characterize Bayesian c-efficient auctions, define the measure $H^\tau\left(p \mid \left\{\left(p^r, \overline{X}^r\right)\right\}_{r=1,\ldots,\tau}\right)$ representing the probability that the dynamic price game ends at price p given the past sequence of aggregate demand and prices. Further, let $\left\{p^1, \ldots, p^M\right\}$ be the set of possible prices that could be the approximate clearing price given the sequence of prices and aggregate demands (we suppress dependence on these values). Finally, let $\Delta(c, M) = \{\tilde{c} \in \mathbb{R} : \exists K \in [M] \text{ such that } c(K) - c(K-1)\}$ be the set of all marginal costs possible given M possible steps. We are now in position to define divide-and-conquer auctions; of which all Bayesian c-efficient auctions are a subset.

Definition D.1. A dynamic price game $(\{\nu^\tau\}, p^0)$ is a **divide-and-conquer auction** when $\nu^\tau(\{\overline{X}^r\}) = \emptyset$ or $\nu^\tau(\{\overline{X}^r\}) = p^{j^*}$ such that

$$j^* \in \left\{ \operatorname*{argmax}_j \left\{ \tilde{c} H^\tau\left(p^j \mid \cdot\right) - \max_{\overline{X}} \left\{ \mathbb{P}\left[\overline{X} = \overline{X}^*(p^j, \theta)\right] - \max_h \left\{ H^{\tau+1}\left(h \mid \left\{\cdot, \left(p^j, \overline{X}\right)\right\}\right)\right\}\right\}\right\} \right.$$
$$\left. \mid \tilde{c} \in \Delta(c, M) \right\}. \tag{11}$$

Theorem D.2. *If a mechanism is Bayesian c-efficient, then it is allocation equivalent to a divide-and-conquer auction.*

Proof. Towards contradiction suppose there exists a Bayesian c-efficient mechanism $\mathcal{M}$ where Equation (11) is violated. As notation, let

$$S(\overline{X}) = \mathbb{P}\left[\overline{X} = \overline{X}^*(p^j, \theta)\right] - \max_h \left\{ H^{\tau+1}\left(h \mid \left\{\cdot, \left(p^j, \overline{X}\right)\right\}\right)\right\}.$$

Let us first consider the case where $\{p^j\}_j$ is the set of all possible clearing prices in the mechanism. Let $\tilde{j}$ be the price selected and let j^* satisfy Equation (11). We will show that

j^* leads to a more c-efficient mechanism by fixing the set of clearing prices queried and then showing that fewer steps (accounting for costs) are taken in expectation. Next observe that, if desired, the size of each region from $p^{\tilde{j}}$ and p^{j^*} could be equalized, no matter the demand and then the dynamic price game could follow the same path. So, for regions that do not include p^{j^*} from $p^{\tilde{j}}$ or vice versa, $S(\overline{X})$ can be equalized. The remaining step is to consider the regions where $p^{\tilde{j}}$ lie. These regions clearly must have at least one more clearing price in them. By the pigeonhole principle, if the prices were already configured in a c-efficient way, then all but at most one price will take at least one more query to reach. Since we are focused on c-efficiency, we can consider just the next query and then induct. So, $S(\overline{X})$ is the correct quantity as this is the additional number of queries that must be performed. Finally, shifting from $p^{\tilde{j}}$ to p^{j^*} shifts the probability the market clears immediately by that probability and leads to a change in cost of taking an additional step to reach $p^{\tilde{j}}$ given the shift. We allow for all possible such costs by allowing j to be selected as long as Equation (11) is satisfied for some $\tilde{c} \in \Delta(c, M)$.

Now consider that for at least some θ, the market does not clear exactly, e.g., $|\Theta| = \infty$ or $c(1)$ is sufficiently large. This reduces Equation (11) to min-maxing the expected probability that the market clears, averaging over all possible demand responses. By the same argument as above, the set in Equation (11) leads to the smallest possible expected search space in the next step and so is optimal given the set of possible clearing prices $\{p^j\}_j$. However, we fixed the set of possible prices to be the same as in $\mathcal{M}$. So, if some other price queries are optimal, then $\mathcal{M}$ is not c-efficient either. $\square$

While Theorem D.2 presents only a necessary condition for Bayesian c-efficient, it is a "tight" condition in the sense that if $\max_k c(k)$ is sufficiently small, for many divide-and-conquer auctions, we can find a prior distribution such that the specified auction is Bayesian c-efficient. By "many," we restrict to auctions such that the potential clearing prices after each step and aggregate demand are consistent with preferences satisfying gross substitutes. The simplest way to do this is to restrict the prior to have the market clear at a finite number of values and take $\max_k c(k)$ small enough that it is efficient to query all possible clearing prices, then any divide-and-conquer auction that has a possibility of clearing the market at all possible prices at each step will be rationalizable.

E Section 6 Appendix

The following definition for an extensive form auction is taken from Li (2017):

Definition E.1. An **extensive form auction** G is defined as the tuple

$$\left(H, \prec, A, \mathcal{A}, P, \{\mathcal{I}_i\}_{i \in \{1,\dots,N\}}\right)$$

such that:

(i) H is a set of histories, along with a binary relation $\prec$ on H that represents precedence. In addition:

  (a) $\prec$ forms a partial order and $(H, \prec)$ forms an arborescence.

(b) There exists an initial history $h_\emptyset = h$ such that there does not exists h' where $h' \prec h$.

(c) The set of terminal histories is $Z \equiv \{h : \neg\exists h \text{ such that } h \prec h'\}$.

(d) The set of immediate successors to h is $\text{succ}(h)$.

(ii) A is the set of possible actions.

(iii) $\mathcal{A} : H \backslash h_\emptyset \to A$ maps histories to the most recent action taken to reach it. In addition:

(a) For all h, $\mathcal{A}(h)$ is one-to-one on $\text{succ}(h)$.

(b) The set of actions available at h is

$$A(h) \equiv \bigcup_{h' \in \text{succ}(h)} \mathcal{A}(h').$$

(iv) $P : H \backslash Z \to \{1, \ldots, N\}$ is the player function for any given non-terminal history.

(v) $\mathcal{I}_i$ is a partition of $\{h : P(h) = i\}$ such that:

(a) $A(h) = A(h')$ when h and h' are in the same cell of the partition, and

(b) $A(h) \cap A(h') = \emptyset$ when h and h' are not in the same cell of the partition.

Proof of Proposition 6.2

Towards contradiction, suppose there exists h^* such that $|A(h^*)| > 2$. We now construct a new game with the same outcomes as G, but with lower costs for every outcome. Informally, we will do this by mapping h^* into a series of binary menus and map choices at h^* onto these menus. Formally, we define (G', σ') in the following manner. First, for all $h \in H$ such that it is not the case that $h^* \succ h$, define $G'(h) = G(h)$. Here we are abusing notation to mean that the game (action sets and information sets) and equilibrium play at h are equivalent between G and G'. Next, define $H' = (H \backslash \{h^*\}) \cup \left\{h_1^*, \ldots, h_{|A(h^*)|-1}^*\right\}$. For all $j \in \{1, \ldots, |A(h^*)| - 1\}$, let $\left|A'(h_j^*)\right| = 2$ and $P'(h_j^*) = P(h^*)$. For $j \in \{2, \ldots, |A(h^*)| - 1\}$, $h_j^* \in \text{succ}'\left(h_{\lfloor j/2 \rfloor}^*\right)$. Label the action set at h^* as $A(h^*) = \left\{a_1, ..., a_{|A(h^*)|}\right\}$. For any history $h \in H$ such that there exists j where $a_j = \mathcal{A}(h)$, we know that $h \in H'$ and we let $h \in \text{succ}'\left(h_{|A(h^*)| - \lceil j/2 \rceil}^*\right)$. We also assume that the information sets are the same at h_j^* and h^*: For all j and $\tilde{h}^*$, if $\mathcal{I}(h^*) = \mathcal{I}(\tilde{h}^*)$, then $\mathcal{I}'(h_j^*) = \mathcal{I}'(\tilde{h}_j^*)$ where $\tilde{h}_j^*$ is defined analogously to h_j^* but for $\tilde{h}^*$.

We now proceed to map the game that takes place after h^* into G'. For all $h \prec h^*$, $G(h) = G'(h)$ except that the information set $\mathcal{I}'(h^*)$ is modified to include the correct partition with regards to $\left\{h_1^*, \ldots, h_{|A(h^*)|-1}^*\right\}$ in G' as opposed to h^* in G. To complete the mapping, we must describe σ' and show that it is a equilibrium in G'. To do this, observe that the same player moves in all h_j^* without any interruptions, and so if $\sigma(h^*, \theta_i) = a_j$ and

$\tilde{h}$ is transitioned to next, then it is an equilibrium to play actions in G' such that $\tilde{h}$ is also transitioned to as outcomes on-path will be the same.

Note that (G, σ) and (G', σ') yield the same outcomes and so to complete the proof we now show that (G', σ') induces weakly less complexity cost for all outcomes and strictly less for certain outcomes. Consider any outcome $z \in Z$ such that it is not the case that $h^* \succ z$, then G and G' are the same and so the costs are equivalent. When $h^* \succ z$, implementing the same outcome under G' is cheaper because the cost at h^* in G is $\tilde{c}\left(|A(h^*)|\right)$; whereas under G', the cost of the constructed menus is at most $\lceil \lg\left(|A(h^*)|\right) \rceil \cdot \tilde{c}(2)$, which is smaller by the super-log property. This means that under $\tilde{c}$-efficiency considerations, (G', σ') is strictly better than (G, σ). □

Proof of Corollary 6.4

Suppose that $\tilde{c}$ is super-log and that (G, σ) is $\tilde{c}$-efficient. By Proposition 6.2, the mechanism is a series of binary choices and by simultaneity, all N participants are asked questions in "rounds". Thus, we know that the $\tilde{c}$-efficient mechanism is a $(\kappa = 1)$-bandwidth mechanism. Clearly, $L/1$ is an integer, so we can apply Theorem 3.2 to conclude that there must be some dynamic price game that implements the same outcome. □

Proof of Corollary 6.5

Observe that when $L = 1$, an L-bandwidth mechanism is equivalent to offering a series of binary choices which the $\tilde{c}$-efficient mechanism must do by Proposition 6.2. When each bidder is queried K times, the total menu cost complexity is $KN\tilde{c}(2)$. Then, we rescale the cost function as menu costs sum across bidders and time costs are assumed not to get the final time cost as $c(K) = KN\tilde{c}(2)$. □

Proof of Corollary 6.6

For any N, assign bidders $\left\{1, \ldots, \lfloor \frac{N}{L} \rfloor\right\}$ to item 1, $\left\{\lfloor \frac{N}{L} \rfloor + 1, \ldots, \lfloor \frac{2N}{L} \rfloor\right\}$ to item 2, etc. Then, we can query the bidders assigned to each object on whether they have the maximum value for that object. Formally, we define

$$s_i^0 = X^0_{\lceil i \cdot \frac{L}{N} \rceil} \text{ and } \nu^0(s^0) = \begin{cases} \emptyset & \sum_{\ell=1}^{L} \mathbb{1}\left[\sum_{i: \lceil i \cdot \frac{L}{N} \rceil = \ell} s_i^0 > 0\right] = L \\ \overline{p} & \text{otherwise} \end{cases}.$$

By the argument in Theorem 5.2, the probability this clears the market can be made arbitrarily large. For any item that does not clear, there is likely another bidder who will desire it at the maximum price. This probability can be made arbitrarily high by having N large enough. So, dividing bidders among items that did not clear and asking if they have the highest value will be optimal in the class of dynamic price games. Note that we are querying the same price multiple times but adjusting s_i^τ. The exact s_i^τ for $\tau > 0$ can be dependent on previous statistics as demand for item ℓ can be informative for demand for item ℓ'. However, since preferences satisfy gross substitutes, if the price of any good decreases, then a higher price for that item will never be queried again. Once we have exhausted all bidders, by the argument in Theorem 5.2, we will decrease the price of all goods that did not clear at the highest price and repeat this procedure. By Corollary 6.4, this is $\tilde{c}$-efficient, as the descending auction just described is the c-efficient dynamic price game.

F Section 7 Appendix

Definition F.1. A dynamic price game $(\{\nu^\tau\}, p^0)$ is an **L-nary auction** if, for all τ and $\left\{\overline{X}^r\right\}_{r=1,\dots,\tau}$, it is either the case that $\nu^\tau\left(\left\{\overline{X}^r\right\}\right) = \emptyset$ or

$$\nu^\tau\left(\left\{\overline{X}^r\right\}\right) \in \underset{p}{\operatorname{argmin}} \left\{ \max_{d \in \{-1,0,1\}^L} d\left(P^\tau\left(p, d, \left\{\overline{X}^r\right\}\right)\right) \right\} \tag{12}$$

where d is the diameter of the set under the L_1 norm:

$$d(S) = \sup_{s,s' \in S} ||s - s'||_1.$$

Proposition F.2. *For any mechanism $\mathcal{M}$, there exists an L-nary auction that is at least as worst-case c-efficient.*

Proof. Towards contradiction, suppose that a mechanism $\mathcal{M}$ is worst-case c-efficient, but is not allocation equivalent to any L-nary auction. By Theorem 3.2, we can restrict to a dynamic price game that offers a price not satisfying Equation (12).

Let us first show that considering $d \in \{-1, 0, 1\}^L$ is a sufficient statistic for aggregate demand. Observe that -1 and 0 capture aggregate demand being 0 and 1 for a good and therefore we only have to show that $d_\ell = 1$ is a sufficient statistic for over-demand. Note that we are focused on exact clearing prices p^* above because we are focused on the worst-case analysis. For any approximate clearing price, we can assume the allocation is such that the lowest value allocation is realized. Thus, the value realized will be the highest willingness to pay for items at the realized allocation (in the case of exact clearing, that is just $\sum_\ell p^*_\ell$). Since the lowest possible valuation will be realized, any aggregate demand of at least 2 for the item will be equivalent.

Suppose the set of potential clearing prices is discrete. If a mechanism is offering a price not satisfying Equation (12), consider the modified mechanism $\mathcal{M}'$ that instead offers a price satisfying Equation (12). Since we are considering the worst case performance and a c-efficient mechanism, the mechanism will never clear until the last possible step. So, the maximum number of possible clearing prices remaining must decrease in size by at least 1 under $\mathcal{M}'$ compared to $\mathcal{M}$. In this case, we can simply add the price offered in some future round and the worst-case depth of the tree would not increase and thus the worst-case complexity would not increase. At the same time, because an additional price was offered the allocation must be weakly more efficient and therefore $\mathcal{M}$ cannot be c-efficient when the set of clearing prices is finite.

Next, we need to show that each demand vector d defines a convex set of possible clearing prices. First, the set of θ which generate any d clearly must be mutually exclusive because everyone is playing the truth-telling equilibrium and a pure strategy. We can assume any tie-breaking rule at the indifference point because the designer gets to select the equilibrium.

Each of these θ generates a clearing price $p^*(\theta)$. The prices on the boundary between regions form a convex hull due to the fact that preferences are gross substitutes. In particular, if two clearing prices have θ such that the same d is generated, then any convex combination of them must also be possible as gross substitutes implies that the valuation function is sub-modular.

When there are a potentially infinite number of clearing prices, let us consider an alternate mechanism where instead of $\mathcal{M}$, every price offered satisfies Equation (12) and the maximum number of steps is the same. Observe that since we are interested in worst-case performance, no price will ever exactly clear the market. So, we are only interested in the approximate performance at the very last step. In this case, since clearing prices vary continuously and the sets $P^\tau(\cdot,\cdot,\cdot)$ are convex, we know that any final price minimizes the diameter of possible clearing prices and is therefore optimal. The diameter is the appropriate metric because whichever allocation is selected, values will be chosen to be as far away as possible which would lead to them being at the furthest possible clearing price. Further, the number of prices queried is the same as $\mathcal{M}$. Therefore, the c-efficiency of the proposed mechanism is weakly higher than $\mathcal{M}$, leading to a contradiction. □

Proof of Proposition 7.3

Let us first prove that the number of queries for an ascending auction is $O(L \cdot M)$. Consider the number of steps needed when $p^* = (\overline{p}_1, \ldots, \overline{p}_L)$ where $\overline{p}_\ell$ is the maximum possible clearing price for item ℓ. Then, because we have assumed that the ascending auction will clear the market at any of the given possible clearing prices, it must be the case that it iterates through all possible clearing prices for each item. To see this, consider an auction where there are two "kinds" of bidders. The first kind has the following preferences: She prefers to consume only good one unless it is at some (heterogeneous) price at which point she may want to buy the second good as well. If that good also has the maximum price, then a bidder may value the third good, etc. through all L goods. These kinds of preferences satisfy gross substitutes since price increases in one dimension weakly increase demand for other goods.[19] The second kind of bidder has single-minded preferences: She has value for a specific item and no value for any other items. She therefore also has the same value for any bundle that includes said item. Observe that as long these single-minded bidders can have value for any item, our set of possible preferences imply that all M^L prices might be clearing prices. Since the clearing prices can be set so that the efficient clearing price must be found no matter the realization of bidders, consider what the ascending auction format must be if all bidders are of the first kind. In this case, the first price must ascend up one price, then the second price has to increase, then the third price, etc. After that, the first price must go up another price and so on until every price is tried. Thus, the ascending auction has $O(L \cdot M)$ number of steps in the worst-case.[20]

Now, let us prove that the number of queries for a simultaneous binary auction is $O(\log M)$. To do this, we will first argue that the number of steps is $O(\log M)$ for additive preferences, and then argue that this provides a bound on the worst-case for other

[19] Note that there is usually no effect on demand for other goods.

[20] In fact, in this case it takes exactly $L \cdot M - 1$ queries.

gross substitute preferences. When preferences are additive, the simultaneous binary auction searches each dimension completely independently. In this case, the simultaneous binary auction splits the search space in half for each dimension ℓ, or the clearing price is exactly known in dimension ℓ. So, the search space is reduced in size by 2^{L_u} each step where L_u is the number of items of which the clearing price is unknown. The total number of clearing prices possible is M^{L_u}. Thus, we can immediately see that the run-time is $O(\log M)$.

The next step is to consider what happens when preferences are not perfectly additive. Here, we will continue to assume that ν^τ is defined as in Equation (3). If preferences are not additive, then two things occur. First, information about demand for one item, provides information about the demand for another. We can ignore this consideration as this implies that the search space is split faster than 2^L. Second, when $d_\ell = 0$, $P^\tau(\cdot, d, \cdot)$ might contain an interior. Observe that $P^\tau(p, d, \cdot)$ must always contain all p^* such that $p^*_\ell = p_\ell$ where $d_\ell = 0$. This is because $P^\tau(p, d, \cdot)$ and $P^\tau(p, -d, \cdot)$ must share a boundary and $d_\ell = 0$ in both cases. But this means that $|P^\tau(p, d, \cdot)|$ must decrease for all d such that $d_\ell \neq 0$ for all ℓ. So, for all such d, the search space is cut by 2^{L_u} where L_u represent the number of uncertain dimensions left. Then, observe that we can also constrain the size of $|P^\tau(p, d, \cdot)|$ where $d_\ell = 0$ for some ℓ by likewise observing that $d_\ell \in \{1, -1\}$ constrains the set because otherwise preferences would not be sub-additive and therefore not gross substitutes. Thus, these sets also must be no more than $M^{L_u}/2^{L_u}$ size. Therefore, the simultaneous binary auction must take $O(\log M)$ steps. We can then apply Proposition F.2 to conclude that the L-nary auction also takes $O(\log M)$ steps as it is c-efficient and distinguishes the same set of prices and so must weakly dominate in terms of number of steps.

The preferences described above prove that the ascending auction is $\Omega(L \cdot M)$ for some set of preferences. We can see that this asymptotic bound is tight on the other auctions by considering the case where we only have to find the clearing price for one good and then the other goods' clearing prices are implied. Of course, this is simpler than the general case as there are only M possible clearing prices. However, even in this case, the best mechanism is the L-nary auction on that single good which has $\Omega(\log M)$ steps. Thus, the L-nary auction and simultaneous binary auction both have $\Theta(\log M)$ number of steps. □