# Shill-Proof Auctions*

Andrew Komo[†] Scott Duke Kominers[‡] Tim Roughgarden[§]

November 13, 2024

**Abstract**

In an auction, a seller may masquerade as one or more bidders in order to manipulate the clearing price. We characterize single-item auction formats that are *shill-proof* in the sense that a profit-maximizing seller has no incentive to submit shill bids. We distinguish between *strong* shill-proofness, in which a seller with full knowledge of bidders' valuations can never profit from shilling, and *weak* shill-proofness, which requires only that the expected equilibrium profit from shilling is nonpositive. The Dutch auction (with a suitable reserve) is the unique (revenue-)optimal and strongly shill-proof auction. Moreover, the Dutch auction (with no reserve) is the unique prior-independent auction that is both efficient and weakly shill-proof. While there are multiple ex-post incentive compatible, weakly shill-proof, and optimal auctions; any optimal auction can satisfy only two properties in the set {static, ex-post incentive compatible, weakly shill-proof}.

*The authors thank Mohammad Akbarpour, Eric Budish, Jeremy Bulow, Peter Cramton, Jacob Leshno, Shengwu Li, Joshua Mollner, Stephen Morris, Marek Pycia, Alex Rees-Jones, Roberto Saitto, Clayton Thomas, and seminar audiences at a16z crypto, MIT, SITE, and Stony Brook for helpful comments. Komo gratefully acknowledges support from the NSF Graduate Research Fellowship Program. Kominers gratefully acknowledges support from the Ng Fund and the Mathematics in Economics Research Fund of the Harvard Center of Mathematical Sciences and Applications. Roughgarden's research at Columbia University is supported in part by NSF awards CCF-2006737 and CNS-2212745, and research awards from the Briger Family Digital Finance Lab and the Center for Digital Finance and Technologies. Part of this work was conducted during the Simons Laufer Mathematical Sciences Institute Fall 2023 program on the Mathematics and Computer Science of Market and Mechanism Design, which was supported by NSF Grant No. DMS-1928930 and by the Alfred P. Sloan Foundation under grant G-2021-16778. Kominers and Roughgarden are members of the Research Team at a16z crypto (for general a16z disclosures, see https://www.a16z.com/disclosures/). Komo was an intern on the a16z crypto Research Team during part of this research. Notwithstanding, the ideas and opinions expressed herein are those of the authors, rather than of a16z or its affiliates. Kominers and Roughgarden also advise companies on marketplace and incentive design, including auction design. Any errors or omissions remain the sole responsibility of the authors.

[†]MIT Economics, Email: komo@mit.edu

[‡]Harvard Business School, Harvard University, and a16z crypto, Email: kominers@fas.harvard.edu

[§]a16z crypto and Columbia University, Email: tim.roughgarden@gmail.com

# 1 Introduction

## 1.1 Shill Bidding in Auctions

**Shill Bidding in Practice.** Auction theory typically assumes that an auction is carried out as described (by the seller or a third party) and focuses solely on the bidders' incentives. Reality is often different. For example, while major auction houses like Christie's or Sotheby's may appear to be carrying out textbook English (ascending) auctions, a degree of skullduggery is often afoot. According to a *New York Times* article from 2000:

> Some tricks of the trade, like an auctioneer's drumming up excitement by acknowledging nonexistent bids only he hears and potential buyers who bid with nearly imperceptible secret signals, have been around for decades. Making up bids, for instance, is known as "bidding off the chandelier" from an era when the grand auction rooms were adorned with ornate lighting.[1]

The practice continues to this day: Christie's Conditions of Sale for their flagship New York location, in a section titled "Auctioneer's Discretion," states (among other things) that "The auctioneer can... move the bidding backwards or forwards in any way he or she may decide...."[2]

Such chandelier bids or *shill* bids—bids submitted by the seller in order to manipulate the final selling price—appear to be particularly common in online auctions. For example, eBay has long gone out of its way to emphasize that shill bidding is forbidden and will be punished:

> We want to maintain a fair marketplace for all our users, and as such, shill bidding is prohibited on eBay... eBay has a number of systems in place to detect and monitor bidding patterns and practices. If we identify any malicious behavior, we'll take steps to prevent it.[3]

According to many eBay users, however, shill bidding remains rampant. Here's a sample quote from the eBay discussion forums:

> The Sellers post a Buy Now price 3–4 times the actual cost of the item. Then they place the item on an auction at $0.01. This to get as many views as possible. The shill comes in shortly after the auction starts and ... is there to prevent the item from being sold below their profit margin.[4]

Chen et al. (2020) find that nearly 10% of all bidders on eBay Motors are shill bidders.

**Shill-Proof Auctions.** Much of auction theory to date encourages truthful bidding through careful auction design, while punting on challenges like seller deviations and collusion via appeal to ummodeled concepts such as the out-of-mechanism enforcement of rules.

[1] See *Genteel Auction Houses Turning Aggressive*, New York Times, April 24, 2000.

[2] See `https://www.christies.com/help/buying-guide-important-information/conditions-of-sale`.

[3] See `https://www.ebay.com/help/policies/selling-policies/selling-practices-policy/shill-bidding-policy?id=4353`.

[4] See `https://community.ebay.com/t5/Buying/My-experience-with-Shill-bidders/td-p/30402514`.

Anecdotes about eBay and other online platforms suggest that such methods are only partially effective at deterring seller deviations. Thus, it makes sense to ask: To what extent can these deviations instead be disincentivized through an auction's design?

The goal of this paper is to understand which auction formats are "shill-proof" in the sense that a seller cannot profit through the submission of shill bids. For private-value auctions, which is our focus here, the reader might well wonder why shill bids matter at all—assuming that the choice of reserve price doesn't affect participation (as it does in the eBay example), isn't a shill bid the same thing as a reserve price?

The answer depends on when the seller has an opportunity to shill and the information available to them at that time. For example, consider an English auction in which the seller also participates as a shill bidder. Suppose the valuations of the (real) bidders are drawn i.i.d. from a regular distribution $F$ and that the opening bid of the auction is set optimally (for revenue), to the monopoly price $\rho^*$ of $F$. As the auction proceeds, with the offered price $p$ starting at $F$ and increasing from there (in increments of $\epsilon$, say), the seller can shill bid at any time. Suppose that the only additional information known to the seller at a given round of the auction is that the remaining bidders are willing to pay at least $p$. Then, the seller asks themself: "now that I know how many bidders are willing to pay at least $p$, do I want to shill and reset the reserve price to $p + \epsilon$?" Under our assumption that $F$ is regular, the answer is "no," and an expected revenue–maximizing seller will never shill.[5,6]

Now suppose that the seller has full knowledge of bidders' realized valuations. In this scenario, the seller will certainly, in some cases, want to shill in an English auction to push the price up to just below the highest of the bidders' valuations. Lest this informational assumption—that the seller knows the full valuation profile—seem impossibly demanding, consider the Dutch (descending) auction (with an arbitrary reserve price). Here, any shill bid by the seller terminates the auction immediately, leaving the seller holding the item and earning zero revenue.

We map out a theory of "shill-proof" auctions, focusing on the following basic questions:

- Which auction formats are "strongly shill-proof" in the sense of the Dutch auction, i.e., with shill bidding being unprofitable even with full knowledge of bidders' realized valuations?
- Which auction formats are "weakly shill-proof" in the sense of the English auction (with bidders' valuations drawn i.i.d. from a regular distribution and an optimally chosen reserve price), i.e., with shill bidding being unprofitable in expectation at equilibrium?
- To what extent are strong and weak shill-proofness compatible with other desirable properties such as optimality, efficiency, ex-post incentive compatiblility, and sealed-bid implementation?

[5] Auction theory experts will now immediately recognize that the English auction with an optimally chosen reserve price is *not* generally shill-proof in this sense when the valuation distribution is not regular.

[6] This doesn't necessarily mean that the auctioneers at Christie's are acting suboptimally, as bidders' valuations in art auctions might be strongly positively correlated—see, for example, Footnote 18 of Milgrom and Weber (1982) for more discussion.

## 1.2 Overview of Results

Iterative auction formats like Dutch and English auctions play a central role in our theory, and accordingly we study (real and shill) bidding in the extensive-form game that is induced by a choice of auction format, relying on a framework for extensive-form auction analysis developed by Li (2017) and Akbarpour and Li (2020). We consider single-item auctions with $N$ bidders. A subset of these are shill bidders, which we model as bidders with zero private value for the item and with utility equal to the seller's revenue.[7] We assume that the private valuations of the non-shill bidders are drawn i.i.d. from a known distribution that is regular (see Definition 2.2). We also assume that shill bidders observe all actions. An auction is then *weakly shill-proof* (Definition 2.3) if there exists an equilibrium of the induced extensive-form game in which the shill bidders never shill (i.e., always bid their true private value of 0). An auction is *strongly shill-proof* (Definition 2.4) if, moreover, shill bidders' equilibrium strategies are ex-post strategies. In our first two results, we focus on *public* auctions (Definition 3.3), meaning auctions in which every bidder's action is publicly observable. This is arguably the most natural model for the analysis of typical iterative auctions such as Dutch and English auctions. Note that a static auction like a sealed-bid first-price auction is not public in this sense.

Next, we summarize the main results of this paper;[8] see also Figure 1.

**Strongly Shill-Proof Auctions.** Our main result (Theorem 3.4) is a uniqueness result for strongly shill-proof auctions: the Dutch auction (with consistent tie-breaking and monopoly reserve price) is strongly shill-proof and optimal (i.e., maximizes the seller's expected revenue), and it is the *only* such auction in the public setting. In particular, strongly shill-proof optimal auctions cannot avoid using a large number of rounds, and they cannot be ex-post incentive compatible (for real bidders). The rough intuition for the proof of this result is that: (i) No matter the information structure, strongly shill-proof auctions must be pay-as-bid (Lemma 3.2); (ii) for any auction format other than a Dutch auction, there exists a history in which some bidder $i$ can indicate that her value is higher than 0 without the auction ending immediately; (iii) optimality in tandem with the public setting then implies that this information effectively induces the auction to revise its reserve price upward or increases perceived competitiveness for the item being sold, which reduces bid shading; and thus (iv) there exist valuations for the bidders such that, if bidder $i$ is a shill bidder, shilling will increase the seller's revenue.

**Weakly Shill-Proof and Efficient Auctions.** Our remaining results concern the richer design space of weakly shill-proof auctions (which contains, at the least, both Dutch and English auctions). We start by investigating *efficient* (and weakly shill-proof) public

[7] The prior literature has sometimes modeled shill bidding via an unknown number of bidders, with some subset of the bidders who end up participating in the auction being shills. Our framework is essentially equivalent: we can take $N$ to be large and require 0 to be in the support of the valuation distribution; and a bidder with value 0 is equivalent (in terms of outcomes) to a bidder not arriving.

[8] We focus on single-item auctions, but our uniqueness results a fortiori provide an upper bound on what is possible for multiunit auctions, as well. More broadly, while our analysis is confined to the context of a single auction, we implicitly bound what is achievable in a sequential auction setting. That said, there are further potential incentives for shill bidding in an auction sequence, such as the desire to have a high clearing price for the first auction in order to drive up attention to later ones; it is not clear that any mechanism design could prevent this type of shill bidding incentive in full generality.

auctions in which, at equilibrium, the item is always awarded to the (real) bidder with the highest valuation. One example of such an auction is a Dutch auction with a reserve price of 0. Interestingly, English auctions are not examples: with a zero reserve, it is not weakly shill-proof (as shill bidders are motivated to push the price up to the monopoly price). But the Dutch auction is not the only efficient and weakly shill-proof auction: beginning with an English auction at the monopoly price and then, should there be no takers, concluding with a Dutch auction (with no reserve) is another example.[9] That this "hybrid" auction format concludes with a Dutch auction is no accident: we prove (in Theorem 3.6) that every robustly weakly shill-proof and efficient auction must conclude with a Dutch auction when all bidders' valuations fall below the monopoly price $\rho^*$ of the distribution.[10] In particular, no ex-post incentive compatible auction can be robustly weakly shill-proof and efficient. It follows that the Dutch auction is the unique prior-independent auction (in the sense of Dhangwatnotai et al. (2015), with no dependence whatsoever on the valuation distribution) that is both efficient and weakly shill-proof (Corollary 3.8).

**Weakly Shill-Proof and Ex-Post Incentive Compatible Optimal Auctions.** The previous two results imply that ex-post incentive compatible auctions cannot be both strongly shill-proof and optimal, nor can they be (robustly) weakly shill-proof and efficient. The English auction (with an optimal reserve price) is, as we've noted, weakly shill-proof, optimal, and ex-post incentive compatible. Is it the unique such auction? Does this combination of properties require a potentially large number of rounds? Our next result (Theorem 4.5) shows that, in general, the answer is no: interrupting an English auction after a sufficiently large number of rounds and closing it out with a second-price auction among the remaining bidders is also weakly shill-proof, optimal, and ex-post incentive compatible. In fact, for any $\varepsilon > 0$, we can always find a valuation distribution such that the number of auction rounds that is required in the worst case to maintain weak shill-proofness is an arbitrarily small fraction of the number of rounds needed in the English auction. (The next result implies that the number of rounds cannot be reduced to 1.)

**Shill-Proof and Ex-Post Incentive Compatible Static Auctions.** Our last result (Theorem 5.2), unlike the others, focuses on *single-action* auctions, meaning auction formats that induce extensive-form games in which each bidder moves exactly once. *Static* auctions are a special case of single-action auctions where each bidder moves simultaneously. Here, in contrast to the prior result, we prove that no single-action auction can simultaneously be weakly shill-proof, optimal, and satisfy even a very weak ex-post incentive compatiblility condition (see Definition 5.1). Thus, an optimal auction can satisfy two and only two of the properties in the set {single-action, ex-post incentive compatible, weakly shill-proof}.[11]

[9] In fact, this auction format closely resembles the Honolulu–Sydney fish auction documented by Hafalir et al. (2023).

[10] Here "robustly" means that the auction specification should depend only on the monopoly price $\rho^*$ of the valuation distribution, and not on its more fine-grained details.

[11] Assuming a regular valuation distribution and a corresponding optimal reserve price, a second-price auction is optimal, single-action, and ex-post incentive compatible; a first-price auction is optimal, single-action, and weakly shill-proof; and an English auction is optimal, ex-post incentive compatible, and weakly shill-proof.

| | Static | Not Static |
|---|---|---|
| Strategy-Proof | Impossible (Theorem 5.2) | Ascending, Screening Auction (Theorem 4.5) |
| Not Strategy-Proof | First-Price Auction | Dutch Auction (Theorem 3.4) |

(a) Weakly shill-proof and optimal auctions

| | Static | Not Static |
|---|---|---|
| Strategy-Proof | Impossible (Theorem 5.2) | Not Robustly (Theorem 3.6) |
| Not Strategy-Proof | Not Robustly (Theorem 3.6) | Dutch Auction (Robustly Unique, Theorem 3.6) |

(b) Weakly shill-proof and efficient auctions

| | Static | Not Static |
|---|---|---|
| Strategy-Proof | Impossible (Theorem 3.4) | Impossible (Theorem 3.4) |
| Not Strategy-Proof | Impossible (Theorem 3.4) | Dutch Auction (Unique, Theorem 3.4) |

(c) Strongly shill-proof and optimal auctions

| | Static | Not Static |
|---|---|---|
| Strategy-Proof | Impossible (Theorem 3.6) | Impossible (Theorem 3.6) |
| Not Strategy-Proof | Impossible (Theorem 3.6) | Dutch Auction (Unique, Theorem 3.6) |

(d) Strongly shill-proof and efficient auctions

Figure 1: **Summary of Results.** Characterization of single-item auction formats that are strongly or weakly shill-proof, along with other properties such as optimality, efficiency, ex-post incentive compatiblility, and sealed-bid implementations.

## 1.3 Related Work

While the idea and practice of shill bidding by a seller have long been well known, the auction theory literature on the topic is surprisingly thin. Chakraborty and Kosmopoulou (2004) consider common value auctions and focus on technological barriers (as opposed to auction formats) that can mitigate shill bidding. Lamy (2009) studies shill bidding specifically in English auctions in which bidders' valuations are affiliated in the sense of Milgrom and Weber (1982), and proves that shill bidding effectively cancels out the effects of affiliation in equilibrium due to real bidders conditioning on bids being fake (see also Izmalkov (2004)). Porter and Shoham (2005) consider a model similar to a second-price auction, motivated by "cheating" by online platforms that can announce a manipulated auction outcome subsequent to collecting all of the bidders' bids. More recently, a number of works (e.g., Basu et al. (2023); Chung and Shi (2023); Lavi et al. (2022); Roughgarden (2021)) have considered shill bidding in the context of blockchain transaction fee mechanism design, with an emphasis on knapsack auctions that are ex-post incentive compatible, shill-proof, and robust to various forms of collusion. Ausubel and Milgrom (2006) and Day and Milgrom (2008) consider shill bids by *bidders* in a multi-item auction, who are looking to exploit complementarities to lower their payments in VCG-type mechanisms—as opposed to shill bids by a seller looking to increase revenue, as is the case of this paper.[12] Contemporary work by Zeng (2024) also studies shill bidding, but takes a different approach to the problem. While we study ex-interim deterrence against shill bidding in extensive form games, he groups auctions into equivalence classes based on the outcome and primarily focuses on ex-ante deterrence against the seller inserting additional bidders into the auction to increase perceived competition. The results in that paper are complementary to ours: While we show that the dynamics of the auction are important for preventing shill bidding, he shows that, when the shill bidders can increase perceived competition outside of taking actions in the auction, only the posted price auction is non-manipulable. See also Footnote 16.

Our theory of shill-proof auctions is similar in spirit to the theory of credible mechanisms

[12] More broadly there is a literature on sybil resistance referred to as "false-name proofness." See, e.g., Conitzer et al. (2010) for a reference.

developed by Akbarpour and Li (2020), and leverages their framework for extensive-form auction analysis. That said, shill-proofness differs conceptually from credibility as shill-proofness focuses on the auctioneer's incentives to insert fake bids whereas credibility focuses on the auctioneer's incentive to truthfully report the actions of a bidder to other bidders. Further, the results in this paper are also qualitatively different. For example, there are a multitude of credible auctions, but only one strongly shill-proof auction and there are a multitude of strategy-proof and weakly shill-proof auctions, but only one strategy-proof and credible auction (see Section 6.1 for more discussion).

More recent research on credible mechanisms, usually with a focus on evading the impossibility results of Akbarpour and Li (2020) under extra assumptions (such as adding cryptographic tools), includes the work of Essaidi et al. (2022), Ferreira and Weinberg (2020), and Chitra et al. (2023). More distantly related papers include that of Haupt and Hitzig (2021), who prove a uniqueness result for the Dutch auction under contextual privacy constraints.

### 1.4 Outline of the Paper

In Section 2, we present the formal model of shill bidding in auctions. Section 3 studies Dutch auctions and their benefits with respect to disincentivizing shill bidding. Section 4 explores which formats are both weakly shill-proofness and ex-post equilibrium for real bidders. Section 5 presents our trilemma result for single-action auctions. And in Section 6, we conclude the paper by discussing extensions.

## 2 Model

In this paper, we consider extensive-form, single item auctions. An extensive-form game $G$ is a tuple of possible histories $H$, and, for each history $h \in H$, functions mapping $h$ to: (i) a player taking an action, $P(h)$; (ii) a set of possible actions, $A(h)$; (iii) an information set,[13] $\mathcal{I}(h)$; and (iv) the most recent action taken, $\mathcal{A}(h)$. As further notation, we denote the starting history of the game by $h_{\varnothing}$ and the set of terminal histories as $Z$; we say $h' \prec h$ if $h'$ precedes $h$, i.e., there exists a sequence of actions that lead from $h'$ to $h$.

We restrict attention to single item auctions, which means that for every terminal history $z \in Z$, we can associate an allocation and transfer vector: $z = (x, t)$, with $\sum_{i=1}^{N} x_i \leqslant 1$ and $x_i \in \{0, 1\}$ for all $i$. As abuse of notation, we will use $x(z), t(z)$ to mean the vectors $(x, t)$ associated with the terminal history $z$. We also assume perfect recall and finite depth. (Definition O.1 in the Online Appendix gives a formal, thorough, and standard definition of extensive form games.) Note that bidders' values for the item are not built into the extensive-form game $G$. Instead, a strategy is a function of both the information set of a bidder and her value for the item. Like most papers in the extensive-form auction literature, we study games with a finite type space because defining auctions with a continuum of types requires defining a general class of continuous-time games. To the authors' knowledge, there is no theory of continuous time games that rivals the generality and flexibility of extensive-form games.

[13]The information sets induce a partition over all possible histories. Players' actions can condition only on $\mathcal{I}(h)$, not on $h$.

## 2.1 Bidders – Real and Shill

In the auction, there is a set of potential bidders $B$, with $|B| = N$, who might participate. Of these potential bidders, a set of real bidders $R$ actually participates. Each bidder $i \in B$ has an independent probability $p$ of participating, $\mathbb{P}[i \in R] = p$.[14] The other bidders, $S = B \backslash R$, are shill bidders whose incentives are completely aligned with the seller/auctioneer's, i.e., their utility is defined by the sum of real bidders' transfers: for $i \in S, u_i(z) = \sum_{j \in R} t_j(z)$. Importantly, the auction $G$ cannot directly condition on the realization of $R$, i.e., the shill bidders are indistinguishable from real bidders during the auction. Each real bidder $i \in R$ has value $v_i$ for the item being sold where $v_i \sim F$ independently for each $i$. Each real bidder has quasi-linear utility: for $i \in R$, $u_i(z) = x_i(z)v_i - t_i(z)$. We assume $F$ is discrete, with support $\mathcal{V}$ consisting of the ordered atoms $0 = v^1 < v^2 < \ldots < v^M$, and we define $f(v^k) = \mathbb{P}_{w \sim F}\left[w = v^k\right]$ to be the pmf of the distribution. As notation, for each shill bidder $i \in S$, we assign $v_i = 0$ and let $v = (v_1, \ldots, v_N)$. The choice of values for shill bidders does not affect their incentives, and by supposing that their values are 0, we can define efficiency and optimality (revenue maximizing) in terms of only $v$ instead of $v$ and $R$.[15] Observe that given how $v$ is generated, we are in the standard, symmetric, single-item independent private values (IPV) setting.

Real bidders have no information about who else is a real bidder.[16] We assume that shill bidders know the set of shill bidders and observe all previous actions taken. Formally, for any history $h$, if $P(h) \in S$, then $\mathcal{I}(h) = \{h\}$. This assumption rules out games with simultaneity (including static games) from the perspective of the shill bidders, but not real bidders.[17] Without cryptography or other unmodeled technologies, we view it as reasonable to assume that while the auction may appear simultaneous to the real bidders, actions are taking place sequentially and the seller can observe those actions.

Our equilibrium concept is pure-strategy Perfect Bayesian Equilibrium; a formal definition of the auction equilibrium $(G, \sigma)$ can be found in Definition A.1. We write $\sigma(v; R)$ for the strategy profile when the value profile is $v$ and the realized set of real bidders is $R$.

## 2.2 Auction Environment

Throughout the paper, we focus on auction equilibria that are ex-post monotone and individually rational: The auction equilibrium $(G, \sigma)$ is **monotone** if, for all $i$, $j$, $v_{-j}$, and $v_j > v'_j$, $\left[t_i\left(\sigma(v; B)\right) > 0 \implies t_i\left(\sigma(v; B)\right) \geqslant t_i\left(\sigma(v'_j, v_{-j}; B)\right)\right]$, and is **individually rational** (IR) if, for all $v$ and $i \in B$, $x_i(\sigma(v; B))v_i - t_i(\sigma(v; B)) \geqslant 0$.

[14] This randomness plays little role in our analysis—we impose it only so that the overarching structure of our model has bidders with ex-ante, symmetric, independent private values. See also Footnote 7.

[15] Here, we assume that the seller has 0 value for the item. Furthermore, when considering optimal auctions, we naturally assume the seller only cares about raising revenue from real bidders.

[16] Unlike the assumption that $N$ is fixed, this assumption is an economically substantive one: The only way for shill bidders to manipulate the outcome of the auction is for shill bidders to take actions in the auction. If the bidders instead knew who were the real bidders, shill bidders could have an incentive to appear as real bidders in order to increase perceived competition. We do not explicitly consider the possibility that real bidders may update about which bidders may be shills over the course of the auction because under our shill-proofness conditions, shill bidders will never make nontrivial bids in equilibrium.

[17] In extensive form games, simultaneity is modeled as the information set of a player having multiple elements.

Because the value distribution is discrete, we must consider what to do if multiple bidders have the same (highest) value. We assume throughout the paper the notion of orderliness introduced by Akbarpour and Li (2020): there exists a fixed priority order—independent of values—over which bidder wins an item if there is a tie.[18]

**Definition 2.1.** An auction equilibrium $(G, \sigma)$ is **orderly** if there exists a total ordering $\rhd$ over $(v_i, i)$ with the following properties:

(i) for all $v, i, j; v_i > v_j \implies (v_i, i) \rhd (v_j, j)$; and

(ii) for all $i, j$, if there exists $m$ such that $(v^m, i) \rhd (v^m, j)$, then for all $k$, $(v^k, i) \rhd (v^k, j)$.

In order to give the optimal auction a well-behaved allocation rule, we suppose the value distribution is regular (with bidders' private values drawn i.i.d.). In Section 5, we relax the regularity assumption. We take our definition of a discrete regular distribution from Elkind (2007):

**Definition 2.2.** A distribution $F$ is **regular**, if for all $k$, the virtual value $\varphi^k = v^k - (v^{k+1} - v^k)\frac{1-F(v^k)}{f(v^k)}$ is non-decreasing.

With regular value distributions, a reserve price $\rho^*$ is optimal if and only if for all $v^k \geqslant \rho^*$, $\varphi^k \geqslant 0$ and for all $v^k < \rho^*$, $\varphi^k < 0$.[19] The direct allocation rule in an orderly, optimal auction is $\tilde{x}_i^*(v) = \mathbb{1}\left\{v_i \geqslant \rho^*, (v_i, i) = \max_{\rhd} \{(v_j, j)\}_{j \in B}\right\}$, and in an orderly, efficient auction the direct allocation rule is $\tilde{x}_i^E(v) = \mathbb{1}\left\{(v_i, i) = \max_{\rhd} \{(v_j, j)\}_{j \in B}\right\}$.

## 2.3 Shill-Proofness

Next, we define our key shill-proofness desiderata. We are interested in auction equilibria in which shill bidders do not shill. Formally, this corresponds to requiring that shill bidders always act like real bidders who have value 0 for the item—since real bidders who have value 0 will never enter non-trivial bids in equilibrium, requiring shill bidders to have the same actions in equilibrium in effect means that shilling does not occur.

**Definition 2.3.** An auction equilibrium $(G, \sigma)$ is **weakly shill-proof** if $\sigma$ is invariant to the realization of $S$, i.e., for all $v$ and $S, S' \subseteq \{i : v_i = 0\}$: $\sigma(v; B \backslash S) = \sigma(v; B \backslash S')$.

Note that Definition 2.3 is a statement about an equilibrium of an auction—it is possible (although we have not found an example of this) that an auction may have both shill-proof equilibria and non-shill-proof equilibria.

We can also strengthen our notion of shill-proofness from not shilling being an equilibrium strategy to being an ex-post strategy:

[18] For example, if ties are broken lexicographically, then the auction is orderly.

[19] There are multiple optimal reserves in our setting, in general, due to the discrete nature of the distribution.

**Definition 2.4.** An auction equilibrium $(G, \sigma)$ is **strongly shill-proof** if it is weakly shill-proof and a ex-post strategy profile for shill bidders, i.e., for all $\sigma'$, $S$, and $v_{-S}$,

$$\sum_{j \in R} t_j(\sigma(0, v_{-S}; R)) \geqslant \sum_{j \in R} t_j\left(\sigma'_S, \sigma_{-S}(0, v_{-S}; R)\right).$$

Strong shill-proofness is obviously preferable to weak shill-proofness (all else equal), especially if there are concerns about a seller somehow acquiring information about real bidders' valuations beyond what is encoded by the prior. As we'll see, however, the design space of weak shill-proof auctions is meaningfully larger than that of strong shill-proof auctions.

## 2.4 Revelation Principle

In order to make progress in understanding shill-proof auction formats, the following revelation principle will be helpful: for every auction equilibrium $(G, \sigma)$, there exists a direct auction that can be summarized by a direct allocation rule $\tilde{x}$, a direct transfer rule $\tilde{t}$, a menu rule $\mu$, and a starting player $\xi_0$.[20] The first input to the menu rule $\mu$ is a set $V$ of valuation profiles of the form $V_1 \times V_2 \times \cdots \times V_N$ with $V_i \subseteq \mathcal{V}$ for all $i$—intuitively, the valuation profiles that are, in equilibrium, consistent with a particular history. The second input is a player $\xi$ who is to move next. The output of the rule is a collection $\left\{\left(W_\ell, \vec{\xi_\ell}\right)\right\}_{\ell \in \{1,\ldots,L\}}$, where the $W_\ell$'s are a partition of $V_\xi$ (from which player $\xi$ will choose one, according to her valuation, the equilibrium strategy $\sigma$ determines the partition) and $\vec{\xi_\ell}$ indicates the next player to move should player $\xi$ choose $W_\ell$. Under $\sigma$, the player $\xi$ will always select the partition $W_\ell$ such that $v_\xi \in W_\ell$. If $\vec{\xi_\ell} = \varnothing$, then the game ends should choice $\ell$ be selected by the bidder $\xi$. For a typical iterative auction, one generally has $\ell = 2$ with the two sets corresponding to types above and below some value, respectively. Or, for a single-action auction, the $W_\ell$'s are generally singletons, with one per type in $V_i$.

We show that for any implementable outcome $(\tilde{x}, \tilde{t})$ of the auction, one can always find a menu rule that is "informative"—the set of possible outcomes differs across partition selections[21]—that also implements the same outcome. So, without loss of generality, we restrict menu rules in this way and then describe an auction equilibrium by $(\tilde{x}, \tilde{t}, \mu, \xi_0)$. (See Lemma A.5 in the Appendix for a more formal treatment.) We refer to $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ as an *auction* when convenient. As is always the case with direct mechanisms, the auction encompasses both the game form and the equilibrium, i.e., by appealing to the revelation principle we have implicitly selected the equilibrium already.

Finally, as notation for later sections, for a set $V = V_1 \times V_2 \times \cdots \times V_N$ of valuation profiles, define $\overline{V}_i = \max\{v_i : v_i \in V_i\}$ to be the maximum possible value of bidder $i$; $\overline{V}_{-i}$ similarly to be the maximum possible value of bidders $j \neq i$; and $\overline{V} = \max_i\left\{\overline{V}_i\right\}$. We define $\underline{V}_i, \underline{V}_{-i}$, and $\underline{V}$ as the corresponding values for minima instead of maxima.

[20] This revelation principle is similar to those found in, for example, Ashlagi and Gonczarowski (2018); Mackenzie (2020); Mackenzie and Zhou (2022); Pycia and Troyan (2023).

[21] This notion of informativeness is very similar to the pruned condition from Akbarpour and Li (2020).

# 3 Dutch Auctions

Define the bidding rule[22]

$$b_i^1(v^m) = v^m - \sum_{k:v^k<v^m} (v^{k+1} - v^k) \frac{\left(F(v^k)\right)^{i-1} \left(F(v^{k-1})\right)^{N-i-1}}{\left(F(v^m)\right)^{i-1} \left(F(v^{m-1})\right)^{N-i-1}}.$$

Then, the Dutch auction is defined as the auction which begins by offering each bidder $i$ the item at $b_i^1(v^M)$, and then if no bidder chooses to buy the item at that price, the item is offered for $b_i^1(v^{M-1})$ and so on until either a bidder has chosen to buy the item or the price to be offered drops below $b_i^1(\rho^*)$, where $\rho^*$ denotes an optimal reserve price. We consider only orderly auctions and therefore, at each price level, bidders are offered the opportunity to buy the item in priority order. Formally:

**Definition 3.1.** The **Dutch auction with reserve price** $\rho^*$ is defined by the optimal allocation rule $\tilde{x}^*$, first-price transfer rule $\tilde{t}^1 = \tilde{x}^* \cdot b^1$, initial player $(\cdot, \xi_0) = \max_{\rhd} \{(0, i)\}$, and menu

$$\mu^D(V, \xi) = \left\{ \left(W_{\mathsf{L}}, \vec{\xi}_{\mathsf{L}}\right), \left(W_{\mathsf{H}}, \vec{\xi}_{\mathsf{H}}\right) \right\},$$

where $W_{\mathsf{H}} = \left\{\overline{V}_\xi\right\}, W_{\mathsf{L}} = V_\xi \backslash \left\{\overline{V}_\xi\right\}, \vec{\xi}_{\mathsf{H}} = \varnothing$, and

$$\vec{\xi}_{\mathsf{L}} = \begin{cases} \left(\cdot, \tilde{\xi}\right) = \max_{\rhd} \left\{\left(\overline{V}_i, i\right) : i \neq \xi\right\} & \exists i \neq \xi \text{ such that } |V_i| > 1 \text{ and } \overline{V}_i \geqslant \rho^* \\ \varnothing & \text{otherwise} \end{cases}.$$

In the following subsections, we explore how the Dutch auction is uniquely suited to preventing shill bidding.

## 3.1 Strongly Shill-Proof Auctions

In this subsection, we first show that all strongly shill-proof auctions must be pay-as-bid and then show that under an assumption that real bidders observe all past actions, we can precisely pin down the Dutch auction as the the only strongly shill-proof auction.

**Lemma 3.2** (Pay-as-bid)**.** *If an optimal auction* $(\tilde{x}^*, \tilde{t}, \mu, \xi_0)$ *is strongly shill-proof, then it must be a pay-as-bid auction. Formally, for all* $\xi, v_\xi$ *and* $v_{-\xi}, v'_{-\xi}$,

$$\tilde{x}^*_\xi \left(v_\xi, v_{-\xi}\right) = \tilde{x}^*_\xi \left(v_\xi, v'_{-\xi}\right) \implies \tilde{t}_\xi \left(v_\xi, v_{-\xi}\right) = \tilde{t}_\xi \left(v_\xi, v'_{-\xi}\right).$$

Observe that Lemma 3.2 (and Theorem 3.4) holds even if we relax the assumption on shill bidders' information sets because an auction being strongly shill-proof means that shill bidders want to report 0 even if they know the precise valuations of other bidders ex-ante. To see why Lemma 3.2 is true, consider the case where $R = \{\xi\}$. Then, the shill bidders will report whichever values maximize $\tilde{t}_\xi$ and so $\tilde{t}_\xi$ must be constant across all outcomes with the same allocation.

To state our main result, we consider public auctions:

[22]See Lemma B.1 in the Appendix for proof that this is the correct form. The difference in transfer function by bidder is because of tie-breaking.

**Definition 3.3.** An auction equilibrium $(G, \sigma)$ is **public** if the information set at any history is all previous actions taken. Formally, for any history $h$, $\mathcal{I}(h) = \{h\}$.

Public auctions are common in practice; from open air fish markets, to auctions on eBay, participants often can see every action other bidders take before choosing what to do.[23] Now that we have restricted the possible information structure, we are in a position to present our main result:

**Theorem 3.4.** *A public and optimal auction is strongly shill-proof if and only if it is the Dutch auction with reserve price $\rho^*$.*

The Dutch auction is strongly shill-proof because any shill bid immediately ends the auction and in that case there would be no transfers from other bidders. To gain an intuition on why uniqueness result holds, we note that the key property of the Dutch auction is that any bid immediately ends the auction. Indeed, for all other auction formats, there exists at least one history such that a bidder can indicate her value is strictly greater than 0 without the auction ending immediately. Such an action has two possible effects: (1) ex-interim increasing the effective reserve price and (2) making it appear as if there is more competition for the item, therefore causing less bid shading. The fact that the auction is public and optimal implies that with a higher reserve price, the transfer from the winner must be higher.[24] We can then conclude that such an auction is not strongly shill-proof because we can consider a valuation vector that generates such a history and have the bidder who can indicate her value is greater than 0 be a shill bidder.

To further understand how the public assumption helps pin down a unique extensive form, please refer to Example O.2 in the Online Appendix for a discussion of single-action, first-price auctions and why they are weakly shill-proof, but not strongly shill-proof.

## 3.2 Weakly Shill-Proof and Efficient Auctions

To dive further into analyzing when a Dutch auction is needed, we now turn to discussing efficient auctions. While the literature has primarily focused on optimal auctions, efficient auctions are important to consider in many cases. One example is two-sided marketplaces, where the auctioneer/market designer and the seller are different entities and may have different objectives. The designer may be interested in allocating goods efficiently while sellers are trying maximize revenue. Our next result shows that in order for an auction to be *weakly* shill-proof and efficient robustly to the prior on the value distribution, part of its game tree must be a Dutch auction. In particular, the auction must be a semi-Dutch auction:

**Definition 3.5.** An efficient auction $(\tilde{x}^E, \tilde{t}, \mu, \xi_0)$ is a **semi-Dutch auction with cutoff** $\rho^*$ if for any $v$ such that $\max_i \{v_i\} < \rho^*$:

[23] Non-examples of public auctions include the FCC spectrum auctions, where bidders typically only learn information on other bidders' actions in rounds (see Milgrom and Segal (2017) for more information).

[24] If the auction were not public, then real bidders' beliefs might not change ex-interim and so their transfers might not change either. For example, in a sealed first-price auction, a shill bidder's actions have no effect on the transfers of other bidders.

(i) $\check{V} = \{w : w < \rho^*\}^N$ is reached and

(ii) $\mu(V, \xi) = \mu^D(V, \xi)$ for any player $\xi$ and possible values $V \subseteq \check{V}$ where $\mu^D$ is the Dutch auction menu rule from Definition 3.1.[25]

By *robust*, we mean that if the auction is not a semi-Dutch auction, then we can find a value distribution such that the auction is not weakly shill-proof. More formally, if the auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ is parameterized by the optimal reserve $\rho^*$, the number of atoms below the reserve $\underline{M}$ and the number of atoms weakly above the reserve $\overline{M}$,[26] the following result holds:

**Theorem 3.6.** *For every public and efficient auction that is not a semi-Dutch auction with cutoff $\rho^*$, there exists a regular value distribution with optimal reserve $\rho^*$ under which the auction is not weakly shill-proof.*

The key step in the proof of Theorem 3.6 resembles the proof of Theorem 3.4—in any non-Dutch auction, shill bidders can ex-interim "raise the reserve price" by changing their actions. However, given that we are interested in weak shill-proofness instead of strong shill-proofness, we have to examine shill bidders' incentives when we take expectations over real bidders' values instead of conditioning directly on their values. Regularity implies that above $\rho^*$, shill bidders do not have an incentive to shill bid in auction formats such as the English auction (see Section 1.1). However, below $\rho^*$, we can always find a regular distribution such that the ex-interim expected value of raising the reserve price is always positive. In the Appendix, we construct the claimed sub-class of regular distributions (see Definition B.3). Informally, the atoms of the value distribution have to be far enough apart so that raising the reserve price a single "level" generates a large amount of additional revenue. So, below $\rho^*$, the auction must resemble the Dutch auction; the class of all such auctions is precisely all semi-Dutch auctions with cutoff $\rho^*$. In the Online Appendix (Example O.3), we provide an example of an auction format that is weakly shill-proof for some value distributions but not others; and the following example presents a real-world setting that roughly fits the premises of Theorem 3.6 where a semi-Dutch auction (that is not a Dutch auction) is used:

**Example 3.7.** The Honolulu-Sydney fish auctions and Istanbul flower auctions documented by Hafalir et al. (2023) blend elements of the Dutch and English auctions: The auction begins at some intermediate price and if anyone bids, then the price ascends like in the English auction. If no one bids, the price descends until someone bids like in the Dutch auction.[27]

[25]Technically, $\mu^D(V, \xi)$ is defined only for optimal auctions; however, it is well-defined for efficient auctions if we redefine $\vec{\xi}_{\mathsf{L}}$ as

$$\vec{\xi}_{\mathsf{L}} = \begin{cases} \left(\cdot, \tilde{\xi}\right) = \max_{\triangleright} \left\{\left(\overline{V}_i, i\right) : i \neq \xi\right\} & \exists i \neq \xi \text{ such that } \overline{V}_i > 0 \\ \varnothing & \text{otherwise} \end{cases}. \tag{1}$$

[26]$\underline{M} + \overline{M} = M$.

[27]Honolulu-Sydney auction, once someone bids, other bidders can counter-bid and raise the price once more. However, in practice there is little counter-bidding. On the theoretical side, in an IPV setting, there exists an equilibrium where there is no counter-bidding. Counter-bidding once the Dutch auction starts is not allowed in the Istanbul flower auction.

The Honolulu-Sydney auction plausibly fits the technical assumptions made in Theorem 3.6: The auctions are public, as they take place in person and all bidders can see other bidders' actions. Market participants are interested in efficient outcomes because the goods are perishable and there are positive disposal costs for the sellers. We do not mean to imply that the Honolulu-Sydney auction was instituted precisely because it is shill-proof, but we highlight it as further evidence that in markets where it is difficult to monitor shill bidding, shill-proof mechanisms often arise.

As a final observation in this section, let us note that if instead of allowing the auction format to treat bids above and below the optimal reserve differently, we instead required the auction format to treat all bids identically, then the only public, efficient, and weakly shill-proof auction is the Dutch auction.

**Corollary 3.8.** *For any public and efficient auction that is not a Dutch auction, there exists a regular value distribution under which the auction is not weakly shill-proof.*

*Proof.* Observe that a semi-Dutch auction with cutoff $\rho^* = v^M$ is simply a Dutch auction. Then, apply Theorem 3.6 for a regular distribution with optimal reserve $\rho^* = v^M$. □

# 4 Weakly Shill-Proof and Strategy-Proof Auctions

We have shown that the only optimal auction in which it is an ex-post strategy for shill bidders not to shill (strong shill-proofness) and an equilibrium for real bidders is the Dutch auction. We now investigate the reverse question: what optimal auctions have an ex-post strategy for real bidders (ex-post incentive compatiblility) and an equilibrium for shill bidders not to shill (weak shill-proofness)? Before we explore that question, let us formally define ex-post incentive compatiblility:

**Definition 4.1.** An auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ is **ex-post incentive compatible** if it is a ex-post strategy for real bidders to report their values truthfully: for all $i$, $v$ and $v_i'$,

$$\tilde{x}(v)v_i - \tilde{t}(v) \geqslant \tilde{x}(v_i', v_{-i})v_i - \tilde{t}(v_i', v_{-i}).$$

An optimal auction is ex-post incentive compatible if and only if has the second-price transfer rule (Akbarpour and Li, 2020, Proposition 8):

$$\tilde{t}_i^2(v) = \tilde{x}_i^*(v) \cdot \max\left\{\rho^*, \text{second-highest value in } \{v_1, \ldots, v_N\}\right\}.$$

Note that if a shill bidder knew the valuations of all other bidders, then shill bidding would turn a second-price auction into a first price auction, which bounds the expected profit for a shill bidder from shilling. So, in order to find a ex-post incentive compatible and weakly shill-proof auction, we must find a menu rule that implements a second-price auction where the expected gain from shill bidding is sufficiently small at shill bidders' information sets. As discussed in Section 1.1, for regular value distributions, one weakly shill-proof, ex-post incentive compatible, and optimal auction is the English auction. We formalize the English auction in our framework as follows:

**Definition 4.2.** The **English auction with reserve price** $\rho^*$ is defined as the auction with the optimal allocation rule $\tilde{x}^*$, second-price transfer rule $\tilde{t}^2$, initial player $(\cdot, \xi_0) = \min_{\triangleright} \{(0, i)\}_{i \in B}$, and menu

$$\mu^E(V, \xi) = \left\{ \left(W_{\mathsf{L}}, \vec{\xi}_{\mathsf{L}}\right), \left(W_{\mathsf{H}}, \vec{\xi}_{\mathsf{H}}\right) \right\},$$

$$\text{where } W_{\mathsf{L}} = \{v \in V_\xi : v < \rho^*\} \cup \left\{\underline{V}_\xi\right\}, W_{\mathsf{H}} = V_\xi \backslash W_{\mathsf{L}},$$

$$\vec{\xi}_{\mathsf{L}} = \vec{\xi}_{\mathsf{H}} = \begin{cases} \left(\cdot, \tilde{\xi}\right) = \min_{\triangleright} \left\{(\underline{V}_i, i) : i \neq \xi, \overline{V}_i = v^M, |V_i| > 1\right\} & \overline{V}_{-\xi} = v^M \\ \varnothing & \text{otherwise} \end{cases}.$$

**Remark 4.3.** The English auction with reserve price $\rho^*$ is weakly shill-proof, ex-post incentive compatible, and optimal. Depending on the information bidders have when taking actions, it can also easily be made strategy-proof. See Section 6.2 for a discussion of how our results hold in dominant strategies instead of ex-post strategies.

The English auction is not the only ex-post incentive compatible and weakly shill-proof auction. While the English auction is used frequently, one drawback is that it is "slow"—each bidder can be queried on their willingness-to-pay on the order of $M$ times. Specifically, let $Q^E(F) = \left|\left\{k : v^k \geqslant \rho^*\right\}\right| - 1$ be the worst-case number of times a bidder must be queried. To explore if there are weakly shill-proof and ex-post incentive compatible auctions that require fewer rounds of communication, we introduce a natural "compression" of the English auction. The ascending, screening auction, comprises the following two phases:

1. An English auction is run from $\rho^*$ to some $v^Y$.

2. If necessary, a second-price auction is then run among players who have not dropped out before the value level of $v^Y$.

**Definition 4.4.** The **ascending, screening auction** with screen level $v^Y$ is defined by the optimal allocation rule $\tilde{x}^*$, second-price transfer rule $\tilde{t}^2$, initial player $(\cdot, \xi_0) = \min_{\triangleright} \left\{\left(\overline{V}_i, i\right)\right\}$, and menu

$$\mu(V, \xi) = \begin{cases} \mu^E(V, \xi) & \exists i \text{ such that } \underline{V}_i < v^Y \text{ and } \overline{V}_i = v^M \\ \left\{ \left( \left\{v^k\right\}, \vec{\xi} \right) \right\}_{k \in \{Y, Y+1, \ldots, M\}} & \text{otherwise} \end{cases},$$

$$\text{where } \left(\cdot, \vec{\xi}\right) = \max_{\triangleright} \left\{(0, i) : |V_i| > 1, \underline{V}_i = v^Y\right\} \text{ and } \vec{\xi}^M = \varnothing.$$

This auction reduces the maximum number of times each bidder can be queried to $Q^{AS,Y}(F) = \left|\left\{k : \rho^* \leqslant v^k \leqslant v^Y\right\}\right| + 1$. Because the transfer rule is $\tilde{t}^2$, the ascending, screening auction is ex-post incentive compatible and optimal.

We use the ascending, screening auction format to explore how fast a weakly shill-proof, ex-post incentive compatible and optimal auction can be, as a function of the underlying value distribution. Our next result shows that the ascending, screening auction can be weakly shill-proof, ex-post incentive compatible, optimal, and take an arbitrarily small fraction of queries as compared to the English auction depending on the value distribution:

**Theorem 4.5.** *For all $\varepsilon > 0$, there exists a value distribution $F$ and screen level $v^Y$ such that $Q^{AS,Y}(F)/Q^E(F) < \varepsilon$ and the ascending, screening auction with screening level $v^Y$ is weakly shill-proof, ex-post incentive compatible, and optimal.*

The ascending, screening auction is orderly and optimal by construction; and is ex-post incentive compatiblebecause the English auction phase and the second-price phase both induce the same (ex-post incentive compatible) allocation and transfer rule. The larger $v^Y$ is, the less that can be extracted in expectation from shill bidding and the more likely it is that a shill bidder will win the item if she shill bids. We provide a sufficient minimum bound on $v^Y$ based on a few moments of a distribution (not its number of atoms), such that for distributions with "thin-enough" right tails—in particular monotone hazard rate distributions— the ascending, screening auction is weakly shill-proof (Lemma C.3). We can then construct a sequence of distributions with increasing numbers of atoms and constant $v^Y$ to complete the proof.

# 5 Single-Action Auctions

Theorem 4.5 shows, by "compressing" an English auction, that there exists a weakly shill-proof and ex-post incentive compatible auction in which, for some value distributions, bidders take far fewer actions than in an English auction. We now show that such compression has its limits, and more generally that there is no weakly shill-proof, ex-post incentive compatible, and optimal auction in which each bidder takes a single action. This impossibility result holds even after relaxing the assumptions that $F$ is regular and after weakening our concept of ex-post incentive compatiblility.

## 5.1 Set-Up

Let us begin by defining a **single-action auction**. An auction is considered single-action when each bidder takes precisely one action in the auction (under all possible histories). More formally, for any $h_N \in Z$, let $h_\varnothing \prec h_1 \prec \dots h_{N-1} \prec h_N$ be the sequence of histories to reach $h_N$. Then, for all $i \in B$, there exists a unique $n \leqslant N$ such that $i = P(h_n)$. Without loss, we label the bidders $1, \dots, N$, in the order that they move and label the action taken by bidder $i$ as $a_i$.[28]

For exposition purposes, instead of tracking the information set $\mathcal{I}_i$ of a bidder $i \in R$, we assume that the information $i$ has when taking an action is a signal $s_i \in \mathcal{S}_i$. This signal is generated via a deterministic function $\psi_i : \left(\bigtimes_{j<i} A_j\right) \to \mathcal{S}_i$ called an **experiment**.[29] For notational convenience, we assume that $\psi_i$ is surjective for all $i$. We can think of the experiment as a garbling of the previous bidders' actions—the experiment can pool together multiple actions from previous bidders to a single signal and so a signal is not always perfectly informative of previous actions. We can recover the public setting with a fully informative

[28]Note that this labeling need not be the same for different histories as the bidder ordering can be endogenous to actions taken.

[29]Abusing notation, we also sometimes take $\psi_i : \mathcal{V}^{i-1} \to \mathcal{S}_i$, i.e., the experiment maps values to signals instead of actions.

experiment, i.e., let $\psi = \text{Id}$, the identity mapping. We can capture classical static game settings via an uninformative experiment that always return the same output, $\psi = \varnothing$. We use $\psi_i^{-1}(s_i)$ to denote the set of $v_{-i}$ that are possible from the perspective of bidder $i$ given its signal.

A revelation principle holds in this setting: for any single-action auction, we can define the direct allocation and transfer rules as $\tilde{x}(v)$ and $\tilde{t}(v)$, respectively, with the appropriate incentive compatibility and individual rationality constraints for real bidders (Lemma D.1 in the Appendix), and appropriate IC constraints for weak and strong shill-proofness (Lemmatta D.2 and D.3, respectively, in the Appendix).

## 5.2 Single-Action, Ex-Post Auctions Cannot Be Shill-Proof

To finish defining all the terms necessary for our main result of this section, we weaken our notion of ex-post incentive compatiblility in the single-action auction setting to ex-post incentive compatiblility for at least a single bidder:[30]

**Definition 5.1.** A single-action auction is **mildly ex-post incentive compatible** if for the associated direct mechanism, there exists a real bidder $i < N$ such that truthfulness is an ex-post strategy conditional on the realization of her signal: there exists bidder $i < N$, such that for all $i, v_i, v_i', s_i$ and $v_{-i}, v'_{-i} \in \psi_i^{-1}(s_i)$, $\tilde{x}_i(v) \cdot v_i - \tilde{t}_i(v) \geqslant \tilde{x}_i(v') \cdot v_i - \tilde{t}_i(v')$.

**Theorem 5.2.** *There exists no single-action, optimal auction that is mildly ex-post incentive compatible and weakly shill-proof.*

*Proof Sketch.* Consider any real bidder $i < N$. By weak shill-proofness, the transfer from bidder $i$, conditional on winning (or losing) the auction, is invariant to the values of bidders who take actions after her (Lemma D.4 in the Appendix). If this were not the case, then if every bidder $j > i$ is a shill bidder, the shill bidders would report the values that would maximize the transfer from the winning bidder. By mild ex-post incentive compatiblility, the transfer from bidder $i$, conditional on winning the auction, is invariant to her value (Lemma D.5 in the Appendix). This is because if there were multiple winning reports with different transfer amounts, only the smallest transfer amount would make truthful reporting of the value an ex-post strategy. So, in every single-action, optimal auction, the transfer from the winning bidder $i$, can depend only on the values reported by bidders before $i$. But, this means that if a bidder has positive utility for winning the item (as would be the case if $v_i > v_j$ for all $j < i$), then she should report $v^M$ to maximize the probability of winning (without changing the transfer paid upon winning). Thus, the auction must treat bidder $i$ as if she reported $v^M$, which violates the allocation rule of an optimal auction. □

[30] We exclude the last bidder who takes an action from our definition because a take-it-or-leave-it offer to that bidder can be optimal and ex-post incentive compatible. Theorem 5.2 would still hold if we instead defined mild ex-post incentive compatiblility to mean ex-post incentive compatible for at least two bidders.

# 6 Discussion

## 6.1 Shill-Proofness vs. Credibility

As discussed in the review of literature, another notion of "cheating" by the auctioneer is that of *in-credibility* introduced by Akbarpour and Li (2020). An auction is credible if a revenue-maximizing auctioneer has no incentive to lie about what other players are doing. That information environment differs from this paper because we assume that bidders correctly (though perhaps not fully) perceive the actions of other players and where in the game tree they are. In the Online Appendix, we formally define credibility in our setting (Definition O.6) and prove the following implications:

**Proposition 6.1.** *Suppose $(G, \sigma)$ is an optimal auction. If the auction is strongly shill-proof, then it must be credible. If the auction is credible, then it must be weakly shill-proof.*

We also define *$\psi$-credibility* for single-action auctions (Definition O.8) as a generalization of credibility allowing for bidders to have exogenous signals (where $\psi$ is our notation from Section 5) about the actions of other bidders as well as additional communication from the auctioneer. In a single-action auction, we define $\psi = \text{Id}$ to mean that $\psi_i(a_{j<i}) = a_{j<i}$, i.e., that the signals reveal the actions of previous bidders. We define $\psi = \varnothing$ to mean the opposite, $\psi_i(a_{j<i}) = \varnothing$. This is the classic static auction setting. We prove that the implications of Proposition 6.1 still hold and give conditions under which credibility coincides with strong and weak shill-proofness:

**Proposition 6.2.** *Suppose $(G, \sigma)$ is a single-action, optimal auction. If the auction is strongly shill-proof, then it must be $\psi$-credible. If it is $(\psi = \varnothing)$-credible, then it is strongly shill-proof. If the auction is $\psi$-credible, then it must be weakly shill-proof. If it is weakly shill-proof, then it is $(\psi = \text{Id})$-credible.*

Proposition 6.2 implies that Theorem 5.2 is a generalization of the credible trilemma (Akbarpour and Li, 2020, Theorem 1).

## 6.2 Dominant Strategies

This paper focuses on ex-post strategies. However, all our results can be extended to dominant strategies as well. To extend Theorem 3.4 from an ex-post strategy not to shill to a dominant strategy is straight-forward: shill bidding in the Dutch auction always leads to 0 revenue, which means it is a weakly dominated strategy, regardless of what other bidders do. Further, since there exist no other auctions besides the Dutch auction that have an ex-post strategy not to shill bid, there can exist no other auction with a dominant strategy not to shill bid.

To extend Theorem 4.5 from an ex-post equilibrium to a dominant strategy equilibrium for real bidders, some care must be taken in considering the information sets of different bidders when they take actions. However, we can provide dominant-strategy equilibria versions of the English and ascending, screening auctions by assuming bidders move simultaneously each round of the English auction, as well in the second-price auction phase of the ascending, screening auction. Theorem 5.2 holds if we were to instead consider dominant strategies for real bidders as dominant strategy incentive compatibility is a stronger condition than ex-post incentive compatibility.

## References


Akbarpour, Mohammad and Shengwu Li (2020): "Credible auctions: A trilemma," *Econometrica*, 88 (2), 425–467. 4, 7, 9, 10, 14, 18, 21, 36, 37

Ashlagi, Itai and Yannai A. Gonczarowski (2018): "Stable matching mechanisms are not obviously strategy-proof," *Journal of Economic Theory*, 177, 405–425. 10

Ausubel, Lawrence M. and Paul Milgrom (2006): *The lovely but lonely Vickrey auction*, 22–26. 6

Bahrani, Maryam, Pranav Garimidi, and Tim Roughgarden (2024): "Centralization in Block Building and Proposer-Builder Separation," ArXiv:2401.12120. 30

Basu, Soumya, David Easley, Maureen O'Hara, and Emin Gün Sirer (2023): "StableFees: A Predictable Fee Market for Cryptocurrencies," *Management Science*, 69 (11), 6508–6524. 6

Chakraborty, Indranil and Georgia Kosmopoulou (2004): "Auctions with shill bidding," *Economic Theory*, 24, 271–287. 6

Chen, Kong-Pin, Ting-Peng Liang, Ted Chang, Yi-chun Liu, Shou-Yung Yin, and Ya-Ting Yu (2020): "How Serious is Shill Bidding in Online Auctions? Evidence from eBay Motors," Working Paper. 2

Chitra, Tarun, Matheus V. X. Ferreira, and Kshitij Kulkarni (2023): "Credible, optimal auctions via blockchains," ArXiv:2301.12532. 7

Chung, Hao and Elaine Shi (2023): "Foundations of Transaction Fee Mechanism Design," in *Proceedings of the 34th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 3856–3899. 6

Conitzer, Vincent, Nicole Immorlica, Joshua Letchford, Kamesh Munagala, and Liad Wagman (2010): "False-name-proofness in social networks," in *Internet and Network Economics: 6th International Workshop, WINE 2010, Stanford, CA, USA, December 13-17, 2010. Proceedings 6*, Springer, 209–221. 6

Day, Robert and Paul Milgrom (2008): "Core-selecting package auctions," *International Journal of Game Theory*, 36, 393–407. 6

Dhangwatnotai, Peerapong, Tim Roughgarden, and Qiqi Yan (2015): "Revenue Maximization with a Single Sample," *Games and Economic Behavior*, 91, 318–333. 5

Elkind, Edith (2007): "Designing and learning optimal finite support auctions," in *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, SIAM, 736–745. 9, 23

Essaidi, Meryem, Matheus V. X. Ferreira, and S. Matthew Weinberg (2022): "Credible, strategyproof, optimal, and bounded expected-round single-item auctions for all distributions," ArXiv:2205.14758. 7

Ferreira, Matheus V. X. and S. Matthew Weinberg (2020): "Credible, truthful, and two-round (optimal) auctions via cryptographic commitments," in *Proceedings of the 21st ACM Conference on Economics and Computation*, 683–712. 7

Hafalir, Isa, Onur Kesten, Katerina Sherstyuk, and Cong Tao (2023): "When Speed is of Essence: Perishable Goods Auctions," Working Paper. 5, 13

Haupt, Andreas and Zoë Hitzig (2021): "Contextually Private Mechanisms," ArXiv:2112.10812. 7

Izmalkov, Sergei (2004): "Shill Bidding and Optimal Auctions," MIT Working Paper. 6

Lamy, Laurent (2009): "The shill bidding effect versus the linkage principle," *Journal of Economic Theory*, 144 (1), 390–413. 6

Lavi, Ron, Or Sattath, and Aviv Zohar (2022): "Redesigning Bitcoin's fee market," *ACM Transactions on Economics and Computation*, 10 (1), Art. 5. 6

Li, Shengwu (2017): "Obviously strategy-proof mechanisms," *American Economic Review*, 107 (11), 3257–3287. 4, 35

Mackenzie, Andrew (2020): "A revelation principle for obviously strategy-proof implementation," *Games and Economic Behavior*, 124, 512–533. 10

Mackenzie, Andrew and Yu Zhou (2022): "Menu mechanisms," *Journal of Economic Theory*, 204, 105511. 10

Milgrom, Paul and Ilya Segal (2017): *Designing the US incentive auction*, Cambridge University Press, 803–812. 12

Milgrom, Paul R. and Robert J. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50 (5), 1089–1122. 3, 6

Porter, Ryan and Yoav Shoham (2005): "On Cheating in Sealed-Bid Auctions," *Decision Support Systems*, 39 (1), 41–54. 6

Pycia, Marek and Peter Troyan (2023): "A theory of simplicity in games and mechanism design," *Econometrica*, 91 (4), 1495–1526. 10

Roughgarden, Tim (2021): "Transaction fee mechanism design," *ACM SIGecom Exchanges*, 19 (1), 52–55. 6

Zeng, Haoyuan (2024): "Identity-Proof Auctions," Working Paper. 6

# A Model (Section 2) Appendix

**Definition A.1.** Consider any set of real bidders $R$ and tuple $(G, \sigma)$. We restrict the set of potential deviations for shill bidders to

$$\Sigma_S = \{\sigma'_S : \forall v_{-S}, \exists v_S \text{ such that } (\sigma'_S, \sigma_{-S}(v_{-S})) = \sigma(v_S, v_{-S}; R = B)\}.$$

Then, the tuple $(G, \sigma)$ is an **auction equilibrium** if for all $i \in R$ and deviating strategies $\sigma'_i$,

$$\mathbb{E}_{v'_{-i},\tilde{R}}\left[u_i\left(\sigma\left(v_i, v'_{-i}; \tilde{R}\right)\right)\right] \geqslant \mathbb{E}_{v'_{-i},\tilde{R}}\left[u_i\left(\sigma'_i\left(v_i, v'_{-i}; \tilde{R}\right), \sigma_{-i}\left(v_i, v'_{-i}; \tilde{R}\right)\right)\right],$$

and for all $\sigma'_S \in \Sigma_S$,

$$\mathbb{E}_{v'_{-S}}\left[u_i\left(\sigma\left(0, v'_{-S}; R\right)\right)\right] \geqslant \mathbb{E}_{v'_{-S}}\left[u_i\left(\sigma'_S\left(0, v'_{-S}; R\right), \sigma_{-S}\left(0, v'_{-S}; R\right)\right)\right].$$

In Definition A.1, we are restricting shill bidders to acting "as-if" they are real bidders by restricting their actions to those of real bidders with some valuation profile. This restriction allows us to move to a direct mechanism where shill-proofness is defined as it being an equilibrium (ex-interim for weak shill-proofness, ex-post for strong shill-proofness) for all shill bidders to report 0. Note that if we were to enlarge the set $\Sigma_S$ to be the set $\hat{\Sigma}_S$ of all strategy profiles, are main results would not change. For Theorems 3.4, 3.6 and 4.5, we are focused on augmented direct games and in such games $\Sigma_S = \hat{\Sigma}_S$. For Theorem 5.2, we know that $\Sigma_S \subset \hat{\Sigma}_S$ and our impossibility result must still hold if the set of possible deviations by shill bidders is larger; thus, the theorem still holds.

**Lemma A.2.** *An optimal auction $(G, \sigma)$ is **winner-paying**: For all $i$ and $v$,*

$$x_i(\sigma(v; B)) = 0 \implies t_i(\sigma(v; B)) = 0.$$

*Proof.* By the ex-post IR constraint, when $x_i(\sigma(v; B)) = 0$, we have $t_i(\sigma(v; B)) \leqslant 0$. It then follows from the optimality that $t_i(\sigma(v; B)) = 0$. To see this, note that for bidder $j \neq i$, equilibrium constraints on bidder $j$ slacken when moving from $t_i < 0$ to $t_i = 0$ and so her play will remain the same. Meanwhile from bidder $i$, the transfer strictly increases moving from $t_i < 0$ to $t_i = 0$. □

Before we state our revelation principle in this context, we recall (with slight modification of notation) a definition and result from Akbarpour and Li (2020) that will be helpful in the proof.

**Definition A.3** (Akbarpour and Li (2020), Definition 2)**.** A game equilibrium $(G, \sigma)$ is **pruned** if, for any history $h$:

(i) There exists $v$ such that $h \preceq z(\sigma(v; B))$.

(ii) If $h \notin Z$, then $|\text{succ}(h)| \geqslant 2$.

(iii) If $h \notin Z$, then for $i = P(h)$, there exists $v_i$,$v_i'$, and $v_{-i}$ such that

  (a) $h \prec z(\sigma(v; B))$,

  (b) $h \prec z(\sigma(v_i', v_{-i}; B))$, and

  (c) $(x, t)(\sigma(v; B)) \neq (x, t)(\sigma(v_i', v_{-i}; B))$.

**Lemma A.4** (Akbarpour and Li (2020), Proposition 1)**.** *If $(G, \sigma)$ is a game equilibrium, then there exists a game equilibrium $(G', \sigma')$ that is pruned and for all $v$, $(x, t)(\sigma(v; B)) = (x', t')(\sigma'(v; B))$.*

**Lemma A.5** (Augmented Revelation Principle)**.** *For every game equilibrium $(G, \sigma)$ there exists an auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$ that meets the following conditions:*

*(i) There exists a direct mechanism $(\tilde{x}, \tilde{t})$: for all $v$, $\tilde{x}(v) = x(\sigma(v; B))$ and $\tilde{t}(v) = t(\sigma(v; B))$.*

*(ii) There exists a choice menu rule $\mu$ that is a function of the potential values $V = V_1 \times \cdots \times V_N$ and bidder $\xi$. This rule has an output of $L \geqslant 2$ choices characterized as $\left\{\left(W_\ell, \vec{\xi}_\ell\right)\right\}_{\ell \in \{1,\ldots,L\}}$ where:*

(a) $\{W_\ell\}_{\ell \in L}$ *forms a partition of* $V_\xi$, $\vec{\xi_\ell} \in (B \cup \{\varnothing\}) \setminus \{\xi\}$, *and* $\vec{\xi} = \varnothing$ *signifies the game has ended.*

(b) *For any* $\ell$ *such that* $\vec{\xi_\ell} \neq \varnothing$, *let* $\hat{V}^\ell = (V_1, \ldots, V_{\xi-1}, W_\ell, V_{\xi+1}, \ldots, V_N)$. *Then, for any such* $\ell$, *there exists* $v_{\xi_\ell}, v'_{\xi_\ell} \in \hat{V}^\ell_{\xi_\ell}$, *and* $v_{-\xi_\ell} \in \hat{V}^\ell_{-\xi_\ell}$ *such that* $(\tilde{x}, \tilde{t})(v_{\xi_\ell}, v_{-\xi_\ell}) \neq (\tilde{x}, \tilde{t})\left(v'_{\xi_\ell}, v_{-\xi_\ell}\right)$. *If* $v_\xi \in W_\ell$, *then the next player in the game is* $\vec{\xi_\ell}$ *and the menu presented to her is* $\mu(\hat{V}^\ell, \vec{\xi_\ell})$.

(c) *If* $\ell$ *is such that* $\vec{\xi_\ell} = \varnothing$, *then for all* $v, v' \in \hat{V}_\ell$, $\left(\tilde{x}, \tilde{t}\right)(v) = \left(\tilde{x}, \tilde{t}\right)(v')$.

(d) *The first player to take an action is* $\xi_0$, *who is presented the menu* $\mu(\mathcal{V}^N, \xi_0)$.

*Proof.* To prove Condition i, we simply construct $(\tilde{x}, \tilde{t})$ by iterating over all possible $v$ and defining $(\tilde{x}, \tilde{t})$ as the outcome of $\sigma(v; B)$ in $G$.

To prove Condition ii, we first observe that by Definition A.1, shill bidders must act "as-if" they were real bidders and that we have restricted to pure strategies. Thus, we can always label actions as classes $\left(W_\ell, \vec{\xi_\ell}\right)$ of a partition of the remaining possible values for the current player $\xi$ and satisfy Condition ii.iia. Condition ii.iic follows from the fact that $G$ is well defined (with each terminal history associated with a single outcome). Condition ii.iid is simply mapping the first player in $G$ to $\xi_0$ and the auctioneer has no information on bidders' values yet. The fact that $L \geqslant 2$ is equivalent to Conditions i and ii of Definition A.3, and Condition iii of Definition A.3 is equivalent to Condition ii.iib here. We can then apply Lemma A.4 to find a game that satisfies these properties. □

# B Dutch Auctions (Section 3) Appendix

## B.1 Strongly Shill-Proof Auctions (Section 3.1) Appendix

**Proof of Lemma 3.2.**

Towards contradiction, suppose there exists a strongly shill-proof auction $(\tilde{x}^*, \tilde{t}, \mu, \xi_0)$, player $\xi$, and values $v_\xi, v_{-\xi}, v'_{-\xi}$ such that $\tilde{x}_\xi(v) = \tilde{x}_\xi(v_\xi, v'_\xi)$, but $\tilde{t}_\xi(v) \neq \tilde{t}_\xi(v_\xi, v'_\xi)$. WLOG, suppose $\tilde{t}_\xi(v) > \tilde{t}_\xi(v_\xi, v'_\xi)$. Now by Lemma A.2, $\xi$ can only have two different transfers if that player wins the item under the allocation. Then, take $R = \{\xi\}$ and by monotonicity, $\tilde{t}_\xi(v) > \tilde{t}_\xi(v_\xi, v'_\xi) \geqslant \tilde{t}(v'_\xi; 0)$ and thus shilling increases revenue and the auction is not strongly shill-proof. □

Given our auction $(\tilde{x}, \tilde{t}, \mu, \xi_0)$, we define $X_i(v_i; V)$ and $T_i(v_i; V)$ to be the ex-interim quantity and transfer rules, respectively, when bidder $i$ has value $v_i$ and the set of potential values for all bidders is $V = V_1 \times \cdots \times V_N$.

**Lemma B.1.** *For every optimal, weakly shill-proof auction* $(\tilde{x}^*, \tilde{t}, \mu, \xi_0)$, *the ex-interim transfer rule for bidder* $i$ *is*

$$T_i(v_i; V) = X_i(v_i; V)v_i - \sum_{m: v^{j_m} < v_i} \left[X_i(v^{j_m}; V) \cdot (v^{j_{m+1}} - v^{j_m})\right],$$

*where* $\{v^{j_m}\}_m$ *are the ordered atoms of* $V_i$.

*Proof.* To prove that $T$ has the claimed form, we will consider a specific non-truthful reporting: if a bidder has value $v^m$, she commits to mis-reporting (selecting partitions) $v^{m'}$ for the rest of the game. We now follow the proof of Theorem 1 of Elkind (2007). Since our direct mechanism is an equilibrium for real bidders, we must have

$$X_i(v^{j_m};V)v^{j_m} - T_i(v^{j_m};V) \geqslant X_i(v^{j_{m-1}};V)v^{j_m} - T_i(v^{j_{m-1}};V), \text{ and}$$
$$X_i(v^{j_{m-1}};V)v^{j_{m-1}} - T_i(v^{j_{m-1}};V) \geqslant X_i(v^{j_m};V)v^{j_{m-1}} - T_i(v^{j_m};V).$$

Defining $U_i$ to be the ex-interim utility for bidder $i$, the preceding expressions become:

$$U_i(v^{j_m};V) \geqslant U_i(v^{j_{m-1}};V) + (v^{j_m} - v^{j_{m-1}})X_i(v^{j_{m-1}};V), \text{ and}$$
$$U_i(v^{j_{m-1}};V) \geqslant U_i(v^{j_m};V) - (v^{j_m} - v^{j_{m-1}})X_i(v^{j_m};V).$$

Thus, $(v^{j_m}-v^{j_{m-1}})X_i(v^{j_{m-1}};V) \leqslant U_i(v^{j_m};V)-U_i(v^{j_{m-1}};V) \leqslant (v^{j_m}-v^{j_{m-1}})X_i(v^{j_m};V)$. Hence, any IC mechanism is such that

$$U_i(v^{j_m};V) = U_i(v^{j_1};V) + \sum_{k=2}^{m}(v^{j_m} - v^{j_{m-1}})\tilde{X}_i(v^{j_m};V)$$
$$\text{where } \tilde{X}_i(v^{j_m};V) \in \left[X_i(v^{j_{m-1}};V), X_i(v^{j_m};V)\right].$$

Therefore, we have that

$$T_i(v^{j_m};V) = X_i(v^{j_m};V)v^{j_m} - U_i(v^{j_1};V) - \sum_{k=2}^{m}(v^{j_m} - v^{j_{m-1}})\tilde{X}_i(v^{j_m};V). \tag{2}$$

By the ex-post IR condition, we have $U_i(v^{j_1};V) \geqslant 0$ for all $V$. So, solving for the optimal transfer rule from Equation (2),

$$T_i^*(v^{j_m};V) = \max_{U_i,\tilde{X}}\left[X_i(v^{j_m};V)v^{j_m} - U_i(v^{j_1};V) - \sum_{k=2}^{m}(v^{j_m} - v^{j_{m-1}})\tilde{X}_i(v^{j_m};V)\right]$$
$$\text{such that } U_i(v^{j_1};V) \geqslant 0 \text{ and } \tilde{X}_i(v^{j_m};V) \in \left[X_i(v^{j_{m-1}};V), X_i(v^{j_m};V)\right].$$

The solution to this maximization is $U_i(v^{j_1};V) = 0, \tilde{X}_i(v^{j_m};V) = X_i(v^{j_{m-1}};V)$. Thus, Equation (2) becomes

$$T_i(v_i;V) = X_i(v_i;V)v_i - \sum_{m:v^{j_m}<v_i}\left[X_i(v^{j_m};V)\cdot(v^{j_{m+1}} - v^{j_m})\right]. \qquad \square$$

For any value choice $(W,\cdot) \in \mu(\cdot,\cdot)$, let us define $\underline{W} = \min_{w\in W}\{w\}$ and $\overline{W} = \max_{w\in W}\{w\}$, respectively.

**Lemma B.2** (Extended Pay-as-Bid)**.** *Consider a strongly shill-proof, public, optimal auction $(\tilde{x}^*, \tilde{t}, \mu, \xi_0)$. Fix $V, \xi$ and consider any $(W, \vec{\xi}) \in \mu(V,\xi)$. If there exists $v, v' \in V$ such that $v_\xi, v'_\xi \in W$ and $\mu(V,\xi)$ is the last action $\xi$ takes, then,*

$$\tilde{x}^*_\xi(v) = \tilde{x}^*_\xi(v') = 1 \implies \tilde{t}_\xi(v) = \tilde{t}_\xi(v') = \frac{T_\xi(\underline{W};V)}{X_\xi(\underline{W};V)},$$

*i.e., transfers are constant conditional on allocation and are pinned down by the ex-interim outcome functions from the lowest type in the partition.*

*Proof.* Since an auction cannot distinguish between values in the same choice set, we can apply Lemma 3.2 to conclude that if $\mu(V, \xi)$ is the last action $\xi$ takes, then $\tilde{t}(v) = \tilde{t}(v')$. To conclude the proof, we note that $\xi$ wins no matter what her value is in $W$ and then apply Lemma A.2 to observe that $T_\xi(\underline{W}; V) = \tilde{t}_\xi(v) \cdot X_\xi(\underline{W}; V)$. □

**Proof of Theorem 3.4.**

We first show that the Dutch auction is a well-defined, i.e., that the stopping rule allows for the auction to be orderly and optimal.[31] We then show that it is strongly shill-proof. Finally, we show that there are no other public, strongly shill-proof, orderly, optimal auctions.

**The Dutch Auction is Orderly, Optimal, and Strongly Shill-Proof.** The Dutch auction quantity and transfer rule are orderly and optimal (as well as ex-post IR and monotone). Indeed, by construction, the next player $\vec{\xi}$ is always the player with the potentially highest value (including for tie-breaking). So, if that player indicates that she is of the highest possible type, the outcome (allocation and transfer) is fully determined and the auction ends. The auction ends once there are no players who could have values weakly greater than $\rho^*$.

We now prove that the Dutch auction is strongly shill-proof. Towards contradiction, suppose not. So, there must exist some $S$, $\xi \in S$, and $V$ such that $\{\overline{V}_\xi\}$ is selected from the menu $\mu(V, \xi)$. But by construction, this means that the auction immediately ends and the good is allocated to the shill bidder who misreported. By Lemma A.2, the revenue from this deviation is 0, which must be weakly less than any other possible transfer.

**Uniqueness.** Towards contradiction, suppose there exists a menu rule $\tilde{\mu} \neq \mu^D$ that is associated with a public, strongly shill-proof, orderly, optimal auction. Therefore, there exists $V$ and $\xi$ such that $\tilde{\mu}(V, \xi) \neq \mu^D(V, \xi)$. Without loss, we will suppose that $V$ is the first time in the game tree that $\tilde{\mu}$ differs from $\mu^D$. Formally, for all $\hat{V} \supsetneq V$, $\tilde{\mu}(V, \xi) = \mu^D(V, \xi)$. We now proceed in cases.

*Case 1 (Different Next Player Choice).* Suppose $\tilde{\mu}(V, \xi) = \left\{ \left(W_{\mathsf{L}}, \tilde{\vec{\xi}}_{\mathsf{L}}\right), \left(W_{\mathsf{H}}, \tilde{\vec{\xi}}_{\mathsf{H}}\right) \right\}$, where $\tilde{\vec{\xi}}_{\mathsf{L}} \neq \vec{\xi}_{\mathsf{L}}$ or $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \vec{\xi}_{\mathsf{H}}$. If $\tilde{\vec{\xi}}_{\mathsf{L}} = \vec{\xi}_{\mathsf{L}}$, then $\tilde{\vec{\xi}}_{\mathsf{H}} = \vec{\xi}_{\mathsf{H}}$ because the outcome is fully resolved once a bidder selects the high partition. So, we need only consider the case where $\tilde{\vec{\xi}}_{\mathsf{L}} \neq \vec{\xi}_{\mathsf{L}}$. By Definition 2.1, $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \varnothing$ (even if the $\xi$ chooses $\{\overline{V}_P\}$). WLOG, we can assume that bidder $b_1$ is called first, followed by bidder $b_2$ with higher priority than $b_1$, (potentially) followed by the remaining bidders. There must exist such a $b_2$ because otherwise the auction calls players in the same order as the Dutch auction, which we assumed was not the case. Let $R = B \backslash \{b_1\}$, and for some $m$, $v_{b_2} = v^m$ and $v_i = v^{m-2}$ for all bidders $i \notin \{b_1, b_2\}$. Taking the expression from Lemma B.1 and dividing both sides by $X_{b_2}$, we get

$$\frac{T_{b_2}(v^m; V)}{X_{b_2}(v^m; V)} = v^m - \sum_{k: v^{j_k} < v^m} \frac{X_{b_2}(v^{j_k}; V)}{X_{b_2}(v^m; V)} \cdot (v^{j_{k+1}} - v^{j_k}) < v^m.$$

(Note that there must be at least one such $k$ in the summation because otherwise $V_{b_2} = \{v^m\}$ and $b_2$ would not take an action.)

[31] The auction is public by definition.

The last choice $b_2$ makes is to select a partition $W$ such that $\underline{W} \geqslant v^{m-2}$. We can therefore apply Lemma B.2 to conclude that the transfer if $b_1$ reports 0 must be $\frac{T_{b_2}(\underline{W};V)}{X_{b_2}(\underline{W};V)} \leqslant \frac{T_{b_2}(v^m;V)}{X_{b_2}(v^m;V)} < v^m$. If bidder $b_1$ instead reports $v^m$, then bidder $b_2$ will win and the revenue will be $v^m$ and so the auction will not be strongly shill-proof. This argument also applies if only the first $n < N$ bidders are chosen in order because of the orderliness assumption.

*Case 2 (Different Partitions).* Suppose there exists $(W, \cdot) \in \tilde{\mu}(V, \xi)$ such that $W \notin \{W_{\mathsf{L}}, W_{\mathsf{H}}\}$. There are now three sub-cases: $\underline{W} \in [\rho^*, \overline{V}_{-\xi})$, $\underline{W} = \overline{V}_{-\xi}$, and $\underline{W} \in (0, \rho^*)$. We need not consider the case where $\underline{W} = 0$ because in that case either we can consider some other choice $W' \notin \{W_{\mathsf{L}}, W_{\mathsf{H}}\}$ or $W = V_\xi$ which would violate Lemma A.5. We need not consider $\underline{W} > \overline{V}_{-\xi}$ because $V$ is the first time $\tilde{\mu}$ differs from $\mu^D$ and for all $V$, $\xi$, and $(\tilde{W}, \cdot) \in \mu^D(V, \xi)$, it is the case that $\underline{\tilde{W}} \leqslant \overline{V}_{-\xi}$.

*Case 2a ($\underline{W} \in [\rho^*, \overline{V}_{-\xi})$).* In this sub-case, there exists $m^*$ such that $\rho^* \leqslant \underline{W}_\ell \leqslant v^{m^*} < \overline{V}_{-\xi}$. Since $V$ is the first time that $\tilde{\mu}$ differs from $\mu^D$, we can suppose there exists $i$ is such that $(v^{m^*}, \xi) \rhd (\rho^*, i)$ because otherwise the outcome would already be resolved or the player rotation would be the only difference (Case 1). Then, suppose bidder $i$ is such that $i \in R$ and $v_i \geqslant v^{m^*+1}$. Take bidder $\xi \in S$ to shill $v^{m^*}$; and for $k \notin \{i, \xi\}$, take $v_k = \underline{V}_k < v^{m^*}$. Therefore, by Lemma B.1, observe that for the last action $i$ takes, her ex-interim transfer must be higher when shill $\xi$ reports $v^{m^*}$ than when she reports 0. Thus by Lemma B.2, when $v^{m^*} < \overline{V}_{-\xi}$, there exists a valuation vector $v$ such that a shill bidder would want to deviate away from reporting 0—and therefore such an auction is not strongly shill-proof.

*Case 2b ($\underline{W} = \overline{V}_{-\xi}$).* In this sub-case, we know $\rho^* \leqslant \underline{W}_\ell = \overline{V}_{-\xi}$. Since $V$ has been generated via a Dutch auction so far, $\xi$ is such that for all $j \neq \xi$, $(\overline{V}_j, j) \rhd (\overline{V}_j, \xi)$, i.e., the current player has the lowest tie-breaking priority. Letting $j \in R$ and $v_j = \overline{V}_{-\xi}$, take bidder $\xi \in S$ to report $\overline{V}_{-\xi}$; and for all $k \notin \{j, \xi\}$, take $v_k = \underline{V}_k < \overline{V}_{-\xi}$. As noted, $(\overline{V}_j, j) \rhd (\overline{V}_j, \xi)$ and so bidder $j$ is allocated the item and not shill bidder $\xi$. Therefore, by the same argument as Case 2a, shill bidding will increase revenue.

*Case 2c ($\underline{W} \in (0, \rho^*)$).* By Condition ii.iib of Lemma A.5, we can observe that $\overline{W} \geqslant \rho^*$. In particular, there exists $j$ and $v_j, v_j' \in V_j$ such that $(v_j, j) \rhd (\rho^*, \xi)$ and $(\overline{W}, \xi) \rhd (v_j', j)$. The second inequality is implied by Lemma 3.2 because there must be some chance that $\xi$ could win the auction in order to affect outcomes. If any bidder is ever offered a choice with $\underline{W} \geqslant \rho^*$, then the previous cases imply that a shill bidder can profitably deviate and so we only have to consider the instances where no such choices are offered. Now, suppose $R = \{j\}$. Suppose all shill bidders play the strategy of selecting the partition $\tilde{W}$ such that $0 < \underline{\tilde{W}} < \rho^*$ if such a choice is available. Let the final move that $j$ takes to be $W^{0,j,\text{last}}, W^{S,j,\text{last}}$ under the shill bidders' strategy of selecting 0 and not, respectively. Similarly, define $V^{0,\text{last}}, V^{S,\text{last}}$ as the possible values and $\tilde{t}_j^0, \tilde{t}_j^S$ as the transfers under these respective strategies. Observe that it is without loss to assume that $v^\omega \equiv \underline{W}^{0,j,\text{last}} \leqslant \underline{W}^{S,j,last}$ because in the latter case the shill bidders are always acting as if they have higher values than in the former case. Next, let $p_i^{m,c} = \mathbb{P}\left[(v^m, j) \rhd (v_i, i) \mid V_i^{c,\text{last}}\right]$ and define $\zeta_{i,m} = \frac{p_i^{m,0} p_i^{\omega,S}}{p_i^{m,S} p_i^{\omega,0}}$. Observe that for all $i \neq j$ and $m \leqslant \omega$, $\zeta_{i,m} \geqslant 1$ with at least one strict inequality because $\overline{V}_i^{S,\text{last}} \geqslant \overline{V}_i^{0,\text{last}}$ for all $i$ with at least one strict inequality. So, by Lemmatta B.1 and B.2,[32]

[32] We may assume that the possible values for $j$ are sequential above the reserve otherwise we could consider $j$ as a shill bidder for some other bidder instead by the cases above.

$$
\begin{aligned}
\tilde{t}_j^S - \tilde{t}_j^0 &\geqslant \frac{T_\xi(v^\omega; V^{S,\text{last}})}{X_\xi(v^\omega; V^{S,\text{last}})} - \frac{T_\xi(v^\omega; V^{0,\text{last}})}{X_\xi(v^\omega; V^{0,\text{last}})} \\
&= \sum_{k: v^k \in [\rho^*, v^\omega)} \left( (v^{k+1} - v^k) \prod_{i \neq j} \frac{p_i^{k,0}}{p_i^{\omega,0}} \right) - \sum_{k: v^k \in [\rho^*, v^\omega)} \left( (v^{k+1} - v^k) \prod_{i \neq j} \frac{p_i^{k,S}}{p_i^{\omega,S}} \right) \\
&= \frac{1}{\prod_{i \neq j} p_i^{\omega,0} p_i^{\omega,S}} \cdot \left( \sum_{k: v^k \in [\rho^*, v^\omega)} (v^{k+1} - v^k) \cdot \left( \prod_{i \neq j} p_i^{k,S} p_i^{\omega,0} (\zeta_{i,k} - 1) \right) \right) > 0.
\end{aligned}
$$

Thus, we have described a profitable shill bidding strategy in this sub-case. □

## B.2 Weakly Shill-Proof and Efficient Auctions (Section 3.2) Appendix

In order to build towards a proof Theorem 3.6, we will prove that for a certain class of value distributions, every *weakly* shill-proof and efficient auction must have part of its game tree be a Dutch auction. Formally, we assume that the value distribution is sparse:

**Definition B.3.** A regular distribution $F$ is **sparse** if for all $k < \rho^*$,

$$
v^k - (v^{k+1} - v^k) \frac{f(v^{k+1})}{f(v^k)} < 0. \tag{3}
$$

A distribution is sparse if the atoms are sufficiently far apart. Sparsity can also be a reasonable assumption if the auctioneer has preferences for the auction to be completed quickly, or otherwise finds it costly to distinguish between values that are close to each other.

**Lemma B.4.** *Consider an efficient auction and suppose $F$ is regular and sparse. Let $R$, $V$ such that $V_i = \{v : v \in [\underline{V}_i, \overline{V}_i]\}$ for all $i \in R$, and consider $(\underline{W}, j)$ such that $\underline{W} < \rho^*, j \notin R$ and for all $i \in R, (\overline{V}_i, i) \rhd (\underline{W}, j)$. Then, for all $\gamma < \underline{W}$,*

$$
\mathbb{E}\left[\sum_{i \in R} \tilde{t}_i(v) \mid V = (V_{-j}, \{\gamma\})\right] < \mathbb{E}\left[\sum_{i \in R} \tilde{t}_i(v) \mid V = (V_{-j}, \{\underline{W}\})\right].
$$

*Thus, the following shilling strategy is profitable compared to always reporting $0$: if there exists $(V, \xi, W)$ such that $(W, \cdot) \in \mu(V, \xi)$ and $\underline{W} \in (0, \rho^*)$, then select $W$. Otherwise, select the partition containing $0$.*

*Proof.* Consider any $i \in R$ and $V$ and let $C = \left(\sum_{v^k \in V_i} f(v^k)\right)^{-1}$. Then, applying Equation (2),

$$\mathbb{E}\left[\tilde{t}_i(v) \mid V\right] + U_i(v^{j_1}; V) = \mathbb{E}\left[T_i(v_i; V)\right] + U_i(v^{j_1}; V) = C\sum_m f(v^{j_m})T_i(v^{j_m}; V)$$

$$= C \sum_{m:v^{j_m}\in V_i} f(v^{j_m})\left(X_i(v^{j_m}; V)v^{j_m} - \sum_{k<m}\left[\tilde{X}_i^k(V)\cdot(v^{j_{k+1}} - v^{j_k})\right]\right)$$

$$= C\left[\sum_{m:v^{j_m}\in V_i} f(v^{j_m})X_i(v^{j_m}; V)v^{j_m} - \sum_{m:v^{j_m}\in V_i}\sum_{k<m} f(v^{j_m})\left[\tilde{X}_i^k(V)\cdot(v^{j_{k+1}} - v^{j_k})\right]\right]$$

$$= C \sum_{m:v^{j_m}\in V_i}\left[v^{j_m}X_i(v^{j_m}; V) - (v^{j_{m+1}} - v^{j_m})\frac{F(\overline{V}_i) - F(v^{j_m})}{f(v^{j_m})}\tilde{X}_i^m(V)\right] f(v^{j_m}).$$

Applying the definition of the efficient allocation rule $\tilde{x}^E$, we know that for $(v^m, i) \rhd (\gamma, j)$ and $(v^m, i) \rhd (\gamma', j)$, we can define $X_i(v^m; V_{-j}) \equiv X_i(v^m; V_{-j}, \{\gamma\}) = X_i(v^m; V_{-j}, \{\gamma'\})$. Note that $\underline{W} \leqslant \min_i \{\overline{V}_i\}$ by assumption and therefore, for $\underline{W} \in (\gamma, \rho^*)$,

$$\mathbb{E}\left[\tilde{t}_i(v) \mid V = (V_{-j}, \{\gamma\})\right] - \mathbb{E}\left[\tilde{t}_i(v) \mid V = (V_{-j}, \{\underline{W}\})\right]$$

$$= C \sum_{m:\gamma\leqslant v^{j_m}<\underline{W}}\left[v^{j_m}X_i(v^{j_m}; V) - (v^{j_{m+1}} - v^{j_m})\frac{F(\overline{V}_i) - F(v^{j_m})}{f(v^{j_m})}\tilde{X}_i^m(V)\right] f(v^{j_m})$$

$$\leqslant C \sum_{m:\gamma\leqslant v^{j_m}<\underline{W}}\left[v^{j_m} - (v^{j_m+1} - v^{j_m})\frac{f(v^{j_m+1})}{f(v^{j_m})}\right] f(v^{j_m})X_i(v^{j_m}; V_{-j}) < 0$$

where the final inequality comes from sparsity. And so,

$$\mathbb{E}\left[\sum_{i\in R}\tilde{t}_i(v) \mid V = (V_{-j}, \{\gamma\})\right] < \mathbb{E}\left[\sum_{i\in R}\tilde{t}_i(v) \mid V = (V_{-j}, \{\underline{W}\})\right],$$

as claimed in the lemma. Thus, committing to misreport as $\underline{W}$ is strictly beneficial compared to any strategy that can only report $\gamma < \underline{W}$. □

**Lemma B.5.** *If $F$ is regular and sparse, then every public, weakly shill-proof, and efficient auction is a semi-Dutch auction with cutoff $\rho^*$.*

*Proof.* Suppose $F$ is regular and sparse. Consider an arbitrary weakly shill-proof and efficient auction, $(\tilde{x}^E, \tilde{t}, \mu, \xi_0)$, and consider any $v$ such that $\max_i \{v_i\} < \rho^*$.

*Proof of (i).* First, we prove that, for any player $\xi$ and set of possible values $V$ such that there exists $(W, \cdot) \in \mu(V, \xi)$ where $0 < \underline{W} < \rho^*$, it is the case that $V \subseteq \check{V}$. Towards contradiction, suppose that there exists a $(\xi, V, W)$ such that $V \nsubseteq \check{V}$, $(W, \cdot) \in \mu(V, \xi)$, and $\underline{W} \in (0, \rho^*)$. Let us suppose $\xi \in S$.

Let us first prove it is without loss to assume $(\xi, V, W)$ is such that for all $i$, $\underline{V}_i = 0$ or $\underline{V}_i \geqslant \rho^*$. If there exists $(\xi, V, W)$ with $i$ such that $\underline{V}_i \in (0, \rho^*)$, let us label that set as $V^K$ and let $V^0 \supset V^1 \supset \ldots \supset V^K$ be the sequence of on-path possible value sets to $V^K$. Let the players called along the path be $\xi^0, \xi^1, \ldots, \xi^K$ and the value partition selected by player $k$ to be $W^k$. Note that $V^0 = \mathcal{V}^N$ is such that for all $i$, $\underline{\tilde{V}}_i = 0$ or $\underline{\tilde{V}}_i \geqslant \rho^*$. So, the set $\mathcal{K} = \{k < K : \underline{W}^k \in (0, \rho^*)\} \neq \varnothing$ and therefore $k^* = \min_{k \in \mathcal{K}} \{k\}$ is well-defined. If $k$ is such that $\underline{W}^k \notin (0, \rho^*)$ and for all $i$, $\underline{V}_i^k \notin (0, \rho^*)$, then it must be the case that for all $i$, $\underline{V}_i^{k+1} \notin (0, \rho^*)$. Since $k^*$ is the first time in the game that a player selects a partition with $\underline{W}^k \in (0, \rho^*)$, it must be the case that for all $i$, $\underline{V}_i^k = 0$ or $\underline{\tilde{V}}_i \geqslant \rho^*$. Since $V^{k^*} \supset V^K$, $V^{k^*} \nsubseteq \check{V}$. Thus, $(\xi^{k^*}, V^{k^*}, W^k)$ is such that $V^{k^*} \nsubseteq \check{V}$, $\left(W^{k^*}, \cdot\right) \in \mu\left(V^{k^*}, \xi^{k^*}\right)$, $\underline{W}^{k^*} \in (0, \rho^*)$, and for all $i$, $\underline{V}_i^{k^*} = 0$ or $\underline{V}_i^{k^*} \geqslant \rho^*$.

So, in order to have $\underline{W} \in (0, \rho^*)$, it must be the case that $0 \in V_\xi$. Thus it is possible for $\xi$ to be a shill bidder while having so far only selected partitions that contain 0. Let $S = \{i : \overline{V}_i < \rho^*\} \cup \{\xi\}$. By assumption that $V \nsubseteq \check{V}$, there must exist $i$ such that $\overline{V}_i \geqslant \rho^*$ and thus we can suppose that $R \neq \varnothing$. By assumption that $\underline{W} \in (0, \rho^*)$, we can suppose that $R$ is such that for all $i \in R$, $\overline{V}_i > \underline{W}$. By Lemma B.4, this would contradict the hypothesis that the auction is weakly shill-proof and so we must have $V \subseteq \check{V}$ when there exists $(W, \cdot) \in \mu(V, \xi)$ such that $0 < \underline{W} < \rho^*$.

*Proof of (ii).* We now prove that for any player $\xi$ and set of possible values $V \subseteq \check{V}$, it is the case that $\mu(V, \xi) = \mu^D(V, \xi)$. Consider any option $(W, \cdot) \in \mu(V, \xi)$. Observe that by Lemma B.4, it is not the case that $0 < \underline{W} < \overline{V}_{-\xi}$. So, $\underline{W} \geqslant \overline{V}_{-\xi}$. Since this is the case for all $V$, it must therefore be true that $\underline{W} = \overline{V}_\xi$. This is because if $\overline{V}_\xi > \underline{V} \geqslant \overline{V}_{-\xi}$, then there must have existed some earlier menu $\left(\tilde{W}, \cdot\right) \in \mu\left(\tilde{V}, \tilde{\xi}\right)$ for which $\underline{\tilde{W}} < \overline{\tilde{V}}_{\tilde{\xi}}$.

So far we have proven that $\mu(V, \xi) = \left\{\left(W_{\mathsf{L}}, \tilde{\vec{\xi}}_{\mathsf{L}}\right), \left(W_{\mathsf{H}}, \tilde{\vec{\xi}}_{\mathsf{H}}\right)\right\}$. To complete the proof, we have to prove that $\tilde{\vec{\xi}}_{\mathsf{L}} = \vec{\xi}_{\mathsf{L}}, \tilde{\vec{\xi}}_{\mathsf{H}} = \vec{\xi}_{\mathsf{H}}$. Towards contradiction, suppose $\tilde{\vec{\xi}}_{\mathsf{L}} \neq \vec{\xi}_{\mathsf{L}}$ or $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \vec{\xi}_{\mathsf{H}}$. If $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \vec{\xi}_{\mathsf{H}}$, then, by Lemma A.5, Condition (ii), there exists $i$ such that $\overline{V}_i = \overline{V}_\xi$, $\left(\overline{V}_i, i\right) \rhd \left(\overline{V}_\xi, \xi\right)$. We can let $R = \{i\}$ and then apply Lemma B.4 to contradict the hypothesis that the auction is weakly shill-proof. If $\tilde{\vec{\xi}}_{\mathsf{L}} \neq \vec{\xi}_{\mathsf{L}}$, then, as argued in the proof of Theorem 3.4, the menu presented to $\tilde{\vec{\xi}}_{\mathsf{L}}$ must not have the auction end immediately, no matter what partition $\tilde{\vec{\xi}}_{\mathsf{L}}$ selects. Thus, our previous argument for the case where $\tilde{\vec{\xi}}_{\mathsf{H}} \neq \vec{\xi}_{\mathsf{H}}$ applies, and we can conclude that $\mu(V, \xi) = \mu^D(V, \xi)$. ☐

### Proof of Theorem 3.6

The statement follows as a corollary of Lemma B.5. Consider any optimal reserve $\rho^*$, $\underline{M}$ atoms below the optimal reserve, and $\overline{M}$ atoms above the optimal reserve. We construct a sparse (and regular) distribution $\tilde{F}$ with optimal reserve $\rho^*$, $\underline{M}$ atoms below the optimal reserve, and $\overline{M}$ atoms above the optimal reserve. To begin, let $\delta$ such that $\underline{M}\delta \leqslant \rho^*$ and $(\underline{M} + 1)\delta > \rho^*$. Then for all $k \leqslant \underline{M}$, let $v^{k+1} - v^k = \delta$ and $\tilde{f}(v^k) = e^{-\lambda(k-2)\delta} - e^{-\lambda(k-1)\delta}$. Note that

$$\tilde{\varphi}^k = v^k - (v^{k+1} - v^k)\frac{1 - \tilde{F}(v^k)}{\tilde{f}(v^k)} = (k-1)\delta - \delta\frac{e^{-\lambda(k-1)\delta}}{e^{-\lambda(k-2)\delta} - e^{-\lambda(k-1)\delta}},$$

and so $\tilde{\varphi}^{k+1} - \tilde{\varphi}^k = k\delta - (k-1)\delta = \delta > 0$; hence, $\tilde{F}$ satisfies the regularity condition for $k \leqslant \underline{M}$.

In order for $\rho^*$ to be an optimal reserve of $\tilde{F}$, it must be the case that for $k^*$ such that $\tilde{\varphi}^{k^*} \geqslant 0$ and $\tilde{\varphi}^{k^*-1} < 0$, it is also the case that $\rho^* \in ((k^*-1)\delta, k^*\delta]$. Such a $k^*$ must be equal to $\lceil \check{k} \rceil$, where $\check{k}\delta - \frac{\delta}{e^{\lambda\delta}-1} = 0$. Thus, $\underline{M} - 1 = k^* = \lceil \frac{1}{e^{\lambda\delta}-1} \rceil$.

In order for $\tilde{F}$ to be sparse, it must satisfy Equation (3), which here simplifies to

$$(k-1)\delta \cdot \left(1 + \frac{e^{-\lambda(k-2)\delta} - e^{-\lambda(k-1)\delta}}{e^{-\lambda(k-1)\delta} - e^{-\lambda(k)\delta}}\right) < k\delta \implies \frac{e^{2\lambda\delta} - 1}{e^{\lambda\delta} - 1} < \frac{k}{k-1}.$$

So, $\tilde{F}$ is sparse if

$$e^{2\lambda\delta} < \frac{2 + (e^{\lambda\delta} - 1)(2(k^* - \check{k}) - 1)}{1 - ((k^* - \check{k}) - 1)(e^{\lambda\delta} - 1)}. \tag{4}$$

Selecting $\lambda$ such that $\underline{M} - 1 = \check{k} = k^*$, Equation (4) is satisfied.

Finally, to finish constructing $\tilde{F}$, we simply select atoms $v^{\underline{M}+2}, \ldots, v^{\underline{M}+\overline{M}}$ and respective probability weights to satisfy

$$\tilde{\varphi}^k \text{ is non-decreasing and } \sum_{k=\underline{M}+2}^{\underline{M}+\overline{M}} \tilde{f}(v^k) = e^{-\lambda(\underline{M}+1)\delta}.$$

This system of constraints has at most $\overline{M}$ constraints and $2(\overline{M} - 1)$ free variables, so the system can be satisfied. Thus, we have constructed a regular and sparse $\tilde{F}$ that has the required values of $\rho^*$,$\underline{M}$, and $\overline{M}$. Then, we can apply Lemma B.5 to conclude the proof.

# C Weakly Shill-Proof and Strategy-Proof Auctions (Section 4) Appendix

**Definition C.1.** Let $F$ be a discrete distribution with ordered atoms $0 = v^1 < \ldots < v^M$ and $\mathcal{F}$ be a continuous distribution with p.d.f. $f$. If $Y_{\mathcal{F}} \sim \mathcal{F}$, then $F$ is a **discrete approximation** of $\mathcal{F}$ when $Y_F \sim F$ is defined as

$$Y_F = \begin{cases} v^1 & Y_{\mathcal{F}} \leqslant v^1 \\ v^k & Y_{\mathcal{F}} \in (v^{k-1}, v^k] \\ v^M & Y_{\mathcal{F}} > v^{M-1} \end{cases}. \tag{5}$$

For such a distribution $F$, let $\overline{\Delta} = \max_k \left\{v^k - v^{k-1}\right\}$. As convention, let $F^{-1}$ be the left pseudo-inverse: $F^{-1}(x) = \max\left\{v^k : x \geqslant F(v^k)\right\}$.

**Definition C.2.** Let $F$ be a discrete approximation of $\mathcal{F}$. The distribution $F$ is a **monotone hazard rate (MHR) distribution** if $\frac{f(w^k)}{1-F(w^k)}$ is monotonically increasing in $k$ and $h(x) = \frac{f(x)}{1-\mathcal{F}(x)}$ is monotonically increasing in $x$.

**Lemma C.3.** *If the value distribution is a discrete MHR distribution $F$, then for all*

$$v^Y \geqslant F^{-1}\left(F(\rho^*) + \max_{1 \leqslant n < N}\left\{\left(\max\left\{1 - \frac{\rho^*}{\rho^* + 2\overline{\Delta}}\left(\frac{f(\rho^*)}{1 - F(\rho^*)}\right)^n, 0\right\}\right)^{1/n}\right\}\right), \tag{6}$$

*the ascending, screening auction with screening level $v^Y$ is a weakly shill-proof, ex-post incentive compatible, and optimal auction.*

**Proof of Lemma C.3.**

**The Ascending, Screening Auction is Orderly and Optimal.** We first prove that the auction is well-defined, orderly, and optimal. The transfer and allocation function are orderly and optimal, so we only have to show that the menu rule can induce this outcome function. Let us examine the English auction phase first. The auction ends if and only if $\overline{V} < v^M$. When that occurs, the auction has determined $v_i$ for all $i$ given $v_i \geqslant \rho^*$. Thus, the outcome is fully determined. In the second-price auction phase, each value weakly greater than $v^Y$ is determined precisely (and there are at least two players with values weakly greater than $v^Y$) and so the outcome rule is determined.

**The Ascending, Screening Auction is Ex-Post Incentive Compatible.** Observe that the definition of ex-post incentive compatiblility (Definition 4.1) is a function solely of the direct mechanism $(\tilde{x}, \tilde{t})$ and not of the menu rule $\mu$. Both the English auction phase and the second-price auction use the same transfer function $\tilde{t}^2$. The entire auction has the optimal allocation rule $\tilde{x}^*$ and so the ascending, screening auction is ex-post incentive compatible.

**The Ascending, Screening Auction is Weakly Shill-Proof.** By assumption that the screen level is at least $\rho^*$, any potential shill bidder will be asked to play at least once in the English auction before being able to play in the second-price auction. If the optimal shill bid is 0 in the first round of the English auction, then the auction is weakly shill-proof because once a bidder reports 0, she "drops out" and does not take another action.

Observe that given the form of the transfer rule, the maximum amount that a bidder $i$ with value $v_i$ would have to pay is $v_i$. Thus, the maximum possible gain in revenue from a shill bidder deviating is at most the difference between the first and second moment of $v$. Next, note that MHR distributions are regular. Regularity implies that if a shill bidder reports a non-zero value in the English auction stage and the auction concludes before reaching the second-price stage, the expected gain must be weakly less than 0. So, when considering the expected gain of misreporting, we can think of the expected gain from manipulating outcomes in the English auction component as at most 0 and can focus on manipulating outcomes in the second-price stage. Therefore, the total gains from misreporting as a shill bidder must be bounded above by the probability that a shill bidder is able to manipulate the outcome of the second-price auction multiplied by the expected difference between the first and second moments of the value distribution conditional on reaching the second-price auction stage.

Let $\mathcal{F}$ be the continuous distribution for which $F$ is a discrete approximation. For an exponential distribution with rate $\lambda$, the expected difference between the first and second moments of $T$ independent draws is $\frac{1}{\lambda}$. The exponential distribution, with its constant hazard rate, has the thickest right tail of any MHR distribution and so has the largest expected difference between its first and second moments (see proof of Theorem 5.1 in Bahrani et al. (2024)). In particular, since we are only interested in value draws above the reserve $\rho^*_{\mathcal{F}}$ and $\mathcal{F}$ has a non-decreasing hazard rate, we can take the rate $\lambda = h(\rho^*_{\mathcal{F}}) = \frac{1}{\rho^*_{\mathcal{F}}}$ and conclude that the maximum difference between the first and second moments of $\mathcal{F}$ must be bounded above by $\rho^*_{\mathcal{F}}$. Recall that $h(\rho^*_{\mathcal{F}}) = \frac{1}{\rho^*_{\mathcal{F}}}$ because $\mathcal{F}$ is regular and $\rho^*_{\mathcal{F}}$ is the optimal reserve of $\mathcal{F}$. Examining Equation (5), we can see that our discrete approximation pools draws from a continuous distribution upwards to atoms and so, if the absolute difference between two samples of the continuous distribution is $\kappa$, the absolute difference between the discrete approximation samples would be at most $\kappa + \overline{\Delta}$. Thus, the maximum possible expected difference between the first and second moments of $F$ conditional on being above the reserve is at most $\rho^*_{\mathcal{F}} + \overline{\Delta}$. Further note that this also implies that $\left|\rho^* - \rho^*_{\mathcal{F}}\right| \leqslant \overline{\Delta}$.

Suppose bidder $i$ is a shill bidder and it is the first time she is taking an action. Then, under the rules of the auction, she has not indicated that her value is greater than $\rho^*$ yet. For any real bidder $j \neq i$, there are two cases: either bidder $j$ has indicated her value is weakly greater than $\rho^*$ $\left(\mathbb{P}\left[v_j < v^Y\right] = F(v^Y) - F(\rho^*)\right)$ or she has not yet taken an action $\left(\mathbb{P}\left[v_j < v^Y\right] = F(v^Y)\right)$. So, if $K \leqslant N$ real bidders have not dropped out yet (i.e., indicated that their value is less than $\rho^*$), then the probability that the auction would continue to the second-price auction is at most $1 - \left(F\left(v^Y\right) - F\left(\rho^*\right)\right)^K$. Therefore, the maximum expected gain for a shill bidder from misreporting in her first action of the English auction phase when $K$ bidders have not dropped is at most

$$\left(1 - \left(F\left(v^Y\right) - F\left(\rho^*\right)\right)^K\right)\left(\rho^*_{\mathcal{F}} + \overline{\Delta}\right). \tag{7}$$

We now turn to bounding the loss from reporting a non-zero value as a shill bidder. If shill bidder $i$ misreports her value as $v^m$ at some point in the English auction phase and then she wins the item without taking another action, then the transfer the seller would have received had $i$ not misreported is at least $\max\left\{\rho^*, v^{m-1}\right\} \geqslant \rho^*$, assuming at least one real bidder has value weakly above the reserve. To bound the probability that a real bidder $j$ would have won the item if not for shill bidder $i$'s misreport, we can consider the probability that bidder $j$ has indicated her value is at least $\underline{V}_j \geqslant \rho^*$. By Definition C.2, the hazard rate of $\mathcal{F}$ is non-decreasing. So,

$$\mathbb{P}\left[v_j \leqslant v^m\right] \geqslant \frac{\sum_{\left\{k:\underline{V}_j \leqslant v^k < v^m\right\}} f(v^k)}{1 - F(\underline{V}_j)} \geqslant \frac{f(\underline{V}_j)}{1 - F(\underline{V}_j)} \geqslant \frac{f(\rho^*)}{1 - F(\rho^*)}.$$

Combining the preceding inequality with our hypothesis that $K$ bidders have not dropped out yet, the expected loss for a shill bidder of misreporting is at least

$$\rho^* \cdot \left(\frac{f(\rho^*)}{1 - F(\rho^*)}\right)^K. \tag{8}$$

We conclude the proof by showing that $v^Y$ satisfying Equation (6) implies that the expected revenue loss from misreporting as a shill is weakly larger than the expected gain. Beginning with Equation (6), we can see that for all $K < N$,

$$\begin{aligned}
v^Y &\geqslant F^{-1}\left(F\left(\rho^*\right) + \max_{1 \leqslant n < N}\left\{\left(\max\left\{1 - \frac{\rho^*}{\rho^* + 2\overline{\Delta}}\left(\frac{f(\rho^*)}{1 - F(\rho^*)}\right)^n, 0\right\}\right)^{1/n}\right\}\right) \\
&\geqslant F^{-1}\left(F\left(\rho^*\right) + \left(\max\left\{1 - \frac{\rho^*}{\rho^* + 2\overline{\Delta}}\left(\frac{f(\rho^*)}{1 - F(\rho^*)}\right)^K, 0\right\}\right)^{1/K}\right) \\
&\geqslant F^{-1}\left(F\left(\rho^*\right) + \left(\max\left\{1 - \frac{\rho^*}{\rho^*_{\mathcal{F}} + \overline{\Delta}}\left(\frac{f(\rho^*)}{1 - F(\rho^*)}\right)^K, 0\right\}\right)^{1/K}\right).
\end{aligned}$$

This implies that

$$\rho^* \cdot \left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^K \geqslant \left(1 - (F(v^Y) - F(\rho^*))^K\right)\left(\rho^*_{\mathcal{F}} + \overline{\Delta}\right).$$

The left-hand side of the preceding equation corresponds to Equation (8), the lower bound on the expected loss from misreporting as a shill bidder, and the right-hand side corresponds to Equation (7), the upper bound on the expected gain from misreporting and thus we have shown that it is equilibrium not to shill when $v^Y$ is sufficiently high. □

**Proof of Theorem 4.5.**

Let $F_m$ be the discrete approximation of the exponential distribution with rate $\lambda = 1$ and atoms at $\{0, 2, \ldots, 2m\}$. Then, using the argument from the proof of Theorem 3.6, $F_m$ is regular and MHR. For all $m > 2$, an optimal reserve is $\rho^* = 4$. Observe that $Y^* = 2$ satisfies Equation (6) because

$$\frac{\rho^*}{\rho^* + 2\overline{\Delta}}\left(\frac{f(\rho^*)}{1-F(\rho^*)}\right)^n = \frac{1}{2}(e^2 - 1)^n > 1 \text{ for all } n \geqslant 1.$$

We apply Lemma C.3 to conclude that the ascending, screening auction with screen level $v^{Y^*}$ is weakly shill-proof, ex-post incentive compatible, and optimal for all $F_m$. Then,

$$Q^{AS,Y^*}(F_m) = N\left(Y^* - \min\left\{k : \rho^* \leqslant v^k\right\} + 2\right) = 2N \text{ and}$$
$$Q^E(F_m) = N\left(m - \min\left\{k : \rho^* \leqslant v^k\right\} + 1\right) = (m-1)N.$$

Thus, $Q^{AS}(F_m, Y^*)/Q^E(F_m) \to 0$ as $m \to \infty$, concluding the proof. □

# D Single-Action Auctions (Section 5) Appendix

**Lemma D.1.** *For any single-action, optimal auction, there exist unique $\tilde{x}^* : \mathcal{V}^N \to \{0,1\}^N$ and $\tilde{t} : \mathcal{V}^N \to \mathbb{R}^N$ such that:*

*(i) (Correspondence) For all $v \in \mathcal{V}^N$, $\tilde{x}^*(v) = x(\sigma(v; B))$ and $\tilde{t}(v) = t(\sigma(v; B))$.*

*(ii) (Individual Rationality) For all $i \in B$ and $v$, $\tilde{x}^*_i(v)v_i - \tilde{t}_i(v) \geqslant 0$.*

*(iii) (Incentive Compatibility) For all $R$, $i \in R$, $v_i$, and $v'_i$,*

$$\mathbb{E}_{v_{-i},\tilde{R}}\left[\tilde{x}^*_i\left(\sigma(v;\tilde{R})\right)v_i - \tilde{t}_i(\sigma(v;\tilde{R})) \mid v_i, s_i = \psi(v_{j<i})\right]$$
$$\geqslant \mathbb{E}_{v_{-i},\tilde{R}}\left[\tilde{x}^*_i\left(\sigma(v'_i, v_{-i};\tilde{R})\right)v_i - \tilde{t}_i\left(\sigma(v'_i, v_{-i};\tilde{R})\right) \mid v_i, s_i = \psi(v_{j<i})\right]. \quad (9)$$

*Proof.* To begin, let us note that we can uniquely define $(\tilde{x}^*, \tilde{t})$ point-wise from $\sigma(v; B)$. The direct allocation rule is uniquely defined as $\tilde{x}^*$ for all optimal auctions. Next, the IR constraint (Condition ii) follows immediately from the ex-post IR condition and our construction of $(\tilde{x}^*, \tilde{t})$. Finally, Equation (9) comes from Definition A.1 and recalling that we restrict shill bidders to actions that could have been taken by real bidders. □

**Lemma D.2.** *If a single-action, optimal auction is weakly shill-proof, then for all $R$, $v_{j<\min S}$,*[33] *and $\{v_i\}_{i\in S}$,*

$$\mathbb{E}_v\left[\sum_{k\in R}\tilde{t}_k\left(\{v_i\}_{i\in S},\{v_i\}_{i\notin S}\right)\mid v_{j<\min S}\right]\leqslant\mathbb{E}_v\left[\sum_{k\in R}\tilde{t}_k\left(0,\{v_i\}_{i\notin S}\right)\mid v_{j<\min S}\right].$$

*Proof.* Towards contradiction, suppose there exists $R$, $v_{j<\min S}$, and $\{v_i\}_{i\in S}$ such that

$$\mathbb{E}_v\left[\sum_{k\in R}\tilde{t}_k\left(\{v_i\}_{i\in S},\{v_i\}_{i\notin S}\right)\mid v_{j<\min S}\right]>\mathbb{E}_v\left[\sum_{k\in R}\tilde{t}_k\left(0,\{v_i\}_{i\notin S}\right)\mid v_{j<\min S}\right].$$

We now prove that the deviation by the coalition $S$ where they report $\{v_i\}_{i\in S}$ is profitable and therefore that the auction is not weakly shill-proof. By assumption, a shill bidder observes actions by all bidders who take actions before her. So, $\{v_i\}_{i\in S}$ can condition on $v_{j<\min S}$ when making decisions. Then, the strategy by $S$ of committing to report $\{v_i\}_{i\in S}$ regardless of what other bidders play after $\min_{i\in S}\{i\}$ must be strictly profitable compared to always reporting 0. Thus, we have found a strategy that does strictly better than always reporting 0: When the values before $\min_{i\in S}\{i\}$ are reported as $v_{j<\min S}$, report $\{v_i\}_{i\in S}$. Otherwise, report 0. This strategy in the direct game immediately translates to a profitable deviation in the auction by Definition A.1 and Lemma D.1 and thus the equilibrium is not weakly shill-proof. □

**Lemma D.3.** *If a single-action, optimal auction is strongly shill-proof, then for all $R$, $i\notin R$, $v_i$, and $v_{-i}$, $\sum_{k\in R}\tilde{t}_k(v_i,v_{-i})\leqslant\sum_{k\in R}\tilde{t}_k(0,v_{-i})$.*

*Proof.* Towards contradiction, suppose that there exists $R$, $i\notin R$, $v_i$, and $v_{-i}$ such that $\sum_{k\in R}\tilde{t}_k(v_i,v_{-i})>\sum_{k\in R}\tilde{t}_k(0,v_{-i})$. That means in the direct game, reporting 0 is not a dominant strategy for shill bidders. This implies, from Condition i of Lemma D.1, that there exists a deviation in the auction such that for some value vectors, the seller raises more revenue. Therefore, the auction is not strongly shill-proof. □

Now, when discussing single-action, optimal auctions, we focus on the direct mechanisms associated to weakly shill-proof auctions and so we will refer to an auction as $(\tilde{x},\tilde{t},\psi)$ without reference to $R$.

**Lemma D.4.** *Suppose a single-action, optimal auction $(\tilde{x}^*,\tilde{t},\psi)$ is weakly shill-proof. Then, for all $i$, $v$, and $v'_{j>i}$, $\left[\tilde{x}^*_i(v)=\tilde{x}^*_i\left(v_{j\leqslant i},v'_{j>i}\right)\implies\tilde{t}_i(v)=\tilde{t}_i\left(v_{j\leqslant i},v'_{j>i}\right)\right]$.*

*Proof.* Towards contradiction, suppose there exists $i$, $v$, and $v'_{j>i}$, such that $\tilde{x}^*_i(v)=\tilde{x}^*_i\left(v_{j\leqslant i},v'_{j>i}\right)$, but $\tilde{t}_i(v;s)>\tilde{t}_i\left(v_{j\leqslant i},v'_{j>i};s\right)$. Because the auction is winner-paying, it must then be the case that $\tilde{x}^*_i(v)=\tilde{x}^*_i\left(v_{j\leqslant i},v'_{j>i}\right)=1$. Let $R=\{1,\dots,i\}$. Then,

$$\mathbb{E}_v\left[\sum_{k\in R}\tilde{t}_k(\{v_i\}_{i\in S},\{v_i\}_{i\notin S})\mid v_{j\leqslant\min S}\right]=\tilde{t}_i(v)>\tilde{t}_i\left(v_{j\leqslant i},v'_{j>i}\right)\geqslant\tilde{t}_i(v_{j\leqslant i},0).$$

This violates Lemma D.2, and so we have reached a contradiction. □

[33] $v_{j<\min S}\equiv\{v_j:j<\min_{i\in S}\{i\}\}$.

**Lemma D.5.** *Suppose a single-action, optimal auction $(\tilde{x}^*, \tilde{t}, \psi)$ is mildly ex-post incentive compatible and weakly shill-proof. Then, there exists $i < N$ such that for all $v_i$, $v_i'$, and $v_{-i} \in \psi_i^{-1}(v_{j<i})$, $\left[\tilde{x}_i^*(v) = \tilde{x}_i^*(v_i', v_{-i}) \implies \tilde{t}_i(v) = \tilde{t}_i(v_i', v_{-i})\right]$.*

*Proof.* Let $i < N, R \ni i, v_i, v_i', v_{-i} \in \psi_i^{-1}(v_{j<i})$ such that $\tilde{x}_i^*(v) = \tilde{x}_i^*(v_i', v_{-i})$. WLOG, suppose $v_i > v_i'$. By monotonicity, $\tilde{t}_i(v) \geqslant \tilde{t}_i(v_i', v_{-i})$. Towards contradiction, suppose $\tilde{t}_i(v) > \tilde{t}_i(v_i', v_{-i})$. By the winner-paying property, $\tilde{t}_i(v) > \tilde{t}_i(v_i', v_{-i})$ implies that $\tilde{x}_i^*(v) = \tilde{x}_i^*(v_i', v_{-i}) = 1$. However, note that $\tilde{t}_i(v) > \tilde{t}_i(v_i', v_{-i})$ would mean that the utility of reporting $v_i'$ would be higher than truthful reporting under true value $v_i$ which would violate the mildly ex-post incentive compatiblility and thus $\tilde{t}_i(v_i', v_{-i}) = \tilde{t}_i(v_i', v_{-i})$. □

**Proof of Theorem 5.2.**

Towards contradiction, suppose such an auction did exist. Fix $i < N, s$ and suppose $v_i < v^M$. Combining Lemmatta D.4 and D.5, we can see that for all $v_i'$ and $v_{-i}, v_{-i}' \in \psi_i^{-1}(v_{j<i})$, $\left[\tilde{x}_i^*(v) = \tilde{x}_i^*(v') \implies \tilde{t}_i(v) = \tilde{t}_i(v')\right]$. So, define $\tilde{t}_i^*$ as the (constant) $\tilde{t}_i(v)$ for all $v$ such that $\tilde{x}_i^*(v) = 1$.

By definition, when $\tilde{x}_i^*(v) = 1$, it must also be the case that $\tilde{x}_i^*(v^M, v_{-i}) = 1$. So, applying the winner-paying property (and suppressing that the expectation is conditioned on $s_i = \psi_i(v_{j<i})$), we have

$$\begin{aligned}
&\mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*\left(v^M, v_{-i}\right) v_i - \tilde{t}_i\left(v^M, v_{-i}\right)\right] \\
&\qquad = \mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i^*(v) = 1\right] + \mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i^*(v) = 0, \tilde{x}_i^*\left(v^M, v_{-i}\right) = 1\right], \quad (10) \\
&\text{and } \mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*(v) v_i - \tilde{t}_i(v)\right] = \mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i^*(v) = 1\right]. \quad (11)
\end{aligned}$$

Taking the difference between Equation (10) and Equation (11), we see that

$$\begin{aligned}
&\mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*\left(v^M, v_{-i}\right) v_i - \tilde{t}_i\left(v^M, v_{-i}\right)\right] - \mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*(v) v_i - \tilde{t}_i(v)\right] \\
&\qquad = \mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i^*(v) = 0, \tilde{x}_i^*\left(v^M, v_{-i}\right) = 1\right] \quad (12)
\end{aligned}$$

Now, by definition $\tilde{x}^*$ is monotone, and by assumption $\tilde{t}$ is monotone. If there exists $v^m$ such that $\mathbb{P}[\tilde{x}_i^*(v^m, v_{-i}) = 1] < \mathbb{P}[\tilde{x}_i^*(v^M, v_{-i}) = 1]$, then for such $m$,

$$\begin{aligned}
\mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*(v^m, v_{-i})v^m - \tilde{t}_i(v^m, v_{-i})\right] &\geqslant \mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*(v^{m-1}, v_{-i})v^m - \tilde{t}_i(v^{m-1}, v_{-i})\right] \\
&> \mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*(v^{m-1}, v_{-i})v^{m-1} - \tilde{t}_i(v^{m-1}, v_{-i})\right] \geqslant 0,
\end{aligned}$$

where the last inequality comes from the IR condition. Thus, the IR constraint does not bind for $v_i = v^m$. Since the good has to be allocated to the highest type, for all $i < N$, there exists $v_{-i}$ such that $\tilde{x}_i^*(v) = 0$ and $\tilde{x}_i^*(v^M, v_{-i}) = 1$. Thus,

$$\mathbb{E}_{v_{-i}}\left[v_i - \tilde{t}_i^* \mid \tilde{x}_i^*(v) = 0, \tilde{x}_i^*\left(v^M, v_{-i}\right) = 1\right] > 0. \quad (13)$$

Combining Equations (12) and (13), we see that

$$\mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*\left(v^M, v_{-i}\right) v_i - \tilde{t}_i\left(v^M, v_{-i}\right)\right] > \mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*(v) v_i - \tilde{t}_i(v)\right]. \quad (14)$$

We then apply the weak shill-proofness condition to simplify Equation (9) to

$$\mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*(v) v_i - \tilde{t}_i(v)\right] \geqslant \mathbb{E}_{v_{-i}}\left[\tilde{x}_i^*(v_i', v_{-i}) v_i - \tilde{t}_i(v_i', v_{-i})\right].$$

Taking $v_i' = v^M$, Equation (14) violates the IC constraint from Lemma D.1—and thus we have reached a contradiction. □

# O Online Appendix

The following definition for an extensive form auction is taken[34] from Li (2017):

**Definition O.1.** An **extensive form auction** $G$ is defined as the tuple $(H, \prec, A, \mathcal{A}, P, \{\mathcal{I}_i\}_{i \in B}, (x, t))$ such that:

(i) $H$ is a set of histories, along with a binary relation $\prec$ on $H$ that represents precedence. In addition:

  (a) $\prec$ forms a partial order and $(H, \prec)$ forms an arborescence.

  (b) There exists an initial history $h_\varnothing = h$ such that there does not exists $h'$ where $h' \prec h$.

  (c) The set of terminal histories is $Z \equiv \{h : \neg \exists h \text{ such that } h \prec h'\}$.

  (d) The set of immediate successors to $h$ is $\text{succ}(h)$.

(ii) $A$ is the set of possible actions.

(iii) $\mathcal{A} : H \backslash h_\varnothing \to A$ maps histories to the most recent action taken to reach it. In addition:

  (a) For all $h$, $\mathcal{A}(h)$ is one-to-one on $\text{succ}(h)$.

  (b) The set of actions available at $h$ is

$$A(h) \equiv \bigcup_{h' \in \text{succ}(h)} \mathcal{A}(h').$$

(iv) $P : H \backslash Z \to B$ is the player function for any given non-terminal history.

(v) $\mathcal{I}_i$ is a partition of $\{h : P(h) = i\}$ such that:

  (a) $A(h) = A(h')$ when $h$ and $h'$ are in the same cell of the partition, and

  (b) $A(h) \cap A(h') = \varnothing$ when $h$ and $h'$ are not in the same cell of the partition.

(vi) For every $z \in Z$, $z = (x, t)$, such that $\sum_{i=1}^{N} x_i \leqslant 1$, $x_i \in [0, 1]$, and $t_i \in \mathbb{R}$.

**Example O.2.** Consider the sealed-bid, first-price (pay-as-bid) optimal auction with equilibrium bids $b^1$. The naïve implementation of allocation and transfer rule $(\tilde{x}^*, \tilde{t}^1)$ in a public auction would be to query each bidder sequentially on what her value is and then have the payment rule be $\tilde{t}^1$, but $\tilde{t}^1$ is not the direct transfer rule of any equilibrium of this game. Indeed, consider the last bidder who takes a move, and label that bidder $N$. If $v_N > \max_{i<N} \{b_i\}$, then the only possible equilibrium bid—and therefore the transfer function—is $\max_{i<N} \{b_i\}$.

If we modify the direct transfer rule to represent the bid that each bidder submits in equilibrium in this sequential form (as one could solve for inductively), the auction would be weakly shill-proof by regularity. In particular, while a shill bid can force later bidders to pay a higher price, the probability that no one will want to pay that higher price outweighs the benefit by regularity. However, such an auction is not strongly shill-proof because, given full knowledge of real bidders' valuations, a shill bidder will be incentivized to bid just below the highest valuation of a subsequent bidder.

[34]We modify the definition to remove notation we do not use and to make it specific to auctions.

**Example O.3.** Let $\mathcal{F}(x) = 1 - e^{-0.1x}$ and let $F_1, F_2$ be discrete approximations of $\mathcal{F}$ with atoms $\{0, 5, 9, 14, 20\}$ and $\{0, 3, 7, 14, 20\}$, respectively. It can be verified that both these distributions are regular and have optimal reserve $\rho^* = 14$. Consider a variant of the efficient Dutch auction (see Equation (1)), with the modification that, if all bidders have indicated values less than 20, then the auction queries bidders from lowest-to highest-priority as to whether their value is at least 9. If no one indicates that their value is at least 9, then the Dutch auction continues. If at least one person does indicate that their value is at least 9, then bidders are queried from lowest- to highest-priority as to whether their value is 14, and the transfer is 14 if at least two people have value 14, and 9 if only one person does. It can be verified that if the value distribution is $F_1$, the auction just described is weakly shill-proof, but if the value distribution is $F_2$, then the auction is not weakly shill-proof. When the value distribution is $F_2$, in expectation, a shill bidder will want to report that her value is 9. In fact, Lemma B.5 (see appendix) implies that if the value distribution is $F_2$, then the auction in this example must be a semi-Dutch auction with cutoff at least 14.

## O.1 Credibility

The following definitions are adapted from Akbarpour and Li (2020) to match our notation and specialized to the auction setting:

For any extensive form game $G$, we can define a messaging game as follows:

1. The auctioneer chooses to:
   (a) Select an outcome and end the game; or
   (b) Go to step 2.
2. The auctioneer chooses some bidder $i \in B$ and sends a message $m = I_i \in \mathcal{I}_i$.
3. Bidder $i$ privately observes message $m = I_i$ and chooses reply $r \in A(I_i)$.
4. The auctioneer privately observes $r$.
5. Go to step 1.

We can now write bidder $i$'s observations in the game as $((m_i^k, r_i^k)_{k=1}^{\tau_i}, \omega_i)$ where $\tau_i$ is the number of observations $i$ has and $\omega_i$ is the information partition over outcomes that $i$ observes. Let $o_i(\sigma_0, \sigma, v)$ be $i$'s observation when the auctioneer plays $\sigma_0$, the bidders play $\sigma$, and the type profile is $v$.

**Definition O.4** (Akbarpour and Li (2020))**.** Let $\sigma_0^G$ be the **rule-following auctioneer strategy**. Formally, $\sigma_0^G$ is defined by the following algorithm: Initialize $\hat{h} := h_\varnothing$. At each step, if $\hat{h} \in z$, end the game and choose outcome $(x, t)(\hat{h})$. Else, contact agent $P(\hat{h})$ and send message $m = \mathcal{I}_{P(\hat{h})}$ such that $P(\hat{h}) \in \mathcal{I}_{P(\hat{h})}$. Upon receiving reply $r$, update $\hat{h}$ to $h$ such that $h \in \text{succ}(\hat{h})$ and $\mathcal{A}(h) = r$, then iterate.

**Definition O.5** (Akbarpour and Li (2020), Definition 3)**.** Given some promised strategy profile $(\sigma_0, \sigma)$, auctioneer strategy $\hat{\sigma}_0$ is safe if, for all agents $i \in B$ and all type profiles $v$, there exists $v'_{-i}$ such that $o_i(\hat{\sigma}_0, \sigma, v) = o_i(\sigma_0, \sigma, (v_i, v'_{-i}))$. We denote by $\Sigma_0^*(\sigma_0, \sigma)$ the set of **safe strategies**.

**Definition O.6** (Akbarpour and Li (2020), Definition 4)**.** $(G, \sigma)$ is **credible** if

$$\sigma_0^G \in \underset{\sigma_0 \in \Sigma_0^*(\sigma_0^G, \sigma)}{\operatorname{argmax}} \left\{ \mathbb{E}_v \left[ \sum_{i \in B} t_i(\sigma_0, \sigma, v) \right] \right\}$$

**Proof of Proposition 6.1**

**Strong Shill-Proofness → Credibility.** We prove the contrapositive: Suppose $(G, \sigma)$ is not credible. Let $\hat{\sigma}_0 \in \Sigma_0^*(\sigma_0^G, \sigma)$ be a profitable and safe deviation by the auctioneer. By Definition O.5, there exists $v$ and $\left\{v'_{-i}\right\}_i$ such that $o_i(\hat{\sigma}_0, \sigma, v) = o_i(\sigma_0, \sigma, (v_i, v'_{-i}))$ for all $i$. By Lemma A.2, only one bidder $i^*$ has $t_{i^*}(\sigma(v_{i^*}, v'_{-i^*})) \neq 0$ and so $t_{i^*}(\sigma(v_{i^*}, v'_{-i^*})) > t_{i^*}(\sigma(v))$ because $\hat{\sigma}_0$ is profitable. Then, let $R = \{i^*\}$, and by ex-post monotonicity,

$$t_{i^*}(\sigma(v_{i^*}, v'_{-i^*})) > t_{i^*}(\sigma(v)) \geqslant t_{i^*}(\sigma(v_{i^*}, 0)),$$

and so the auction is not strongly shill-proof.

**Credibility → Weak Shill-Proofness.** We prove the contrapositive: Suppose $(G, \sigma)$ is not weakly shill-proof. Let $\hat{\sigma} \in \Sigma_S$ be a profitable shilling strategy. Then, by Definition A.1, for all $R$, there exists $v, v'$ such that $\hat{\sigma}(v; R) = \sigma(v; B)$. Consider the following reporting strategy $\hat{\sigma}_0$: for all $i \in R$, report play as if $i \in R$ is following $\hat{\sigma}$; and for all $i \notin R$, report in the rule-following manner. This strategy is safe because $\hat{\sigma} \in \Sigma_S$. To see that it is profitable compared to $\sigma_0^G$, consider what happens when the winning bidder $i$ is or is not in $R$.[35] Conditional on $i \in R$ winning, $\hat{\sigma}_0$ increases expected revenue because $\hat{\sigma}$ is a profitable shill bidding strategy. Conditional on $i \notin R$ winning, shill bidding would have led 0 revenue for the seller and, by Lemma A.2, $\hat{\sigma}_0$ must have non-negative revenue. Thus, our described reporting strategy is a profitable, safe strategy and therefore the auction is not credible. □

## O.2 Generalizing Credibility in the Single-Action Case

**Definition O.7.** Fix a single-action, optimal auction with exogenous signal $\psi$ and a set of real bidders $R$. The set of safe deviations to report to $i \in B$ is

$$\mathcal{A}_i^{\psi}(v_{j \leqslant i}) = \\ \left\{a : \exists \tilde{v}_{-i} \text{ such that } [j < i, v_j \neq 0 \implies \tilde{v}_j = v_j] \text{ and } a = (\sigma_i(v_i; \psi_i(\tilde{v}_{j<i})), \sigma_{-i}(v_i, \tilde{v}_{-i}; B))\right\}.$$

The total set of **safe deviations** is

$$\mathcal{A}^{\psi}(v) = \left\{ \left\{a^{\rightsquigarrow i}\right\} : a^{\rightsquigarrow i} \in \mathcal{A}_i^{\psi}(v_{j \leqslant i}) \text{ and } \sum_{i=1}^{N} x_i\left(a^{\rightsquigarrow i}\right) \leqslant 1 \right\}.$$

Definition O.7 allows the auctioneer to report any value she chooses when a bidder's declared valuation is 0. Note that if we take the canonical setting where there are no exogenous signals, the above assumption is without loss.

[35] Note that if no one has value above the optimal reserve, there will be no winner under any safe strategy, so let us only consider the case where the good is allocated.

**Definition O.8.** A single-action, optimal auction is $\psi$**-credible** if for all $v$ and $\{a^{\rightsquigarrow i}\} \in \mathcal{A}^{\psi}(v)$, we have

$$\sum_i t_i(a^{\rightsquigarrow i}) \leqslant \sum_i t_i(\sigma(v; B)).$$

**Lemma O.9.** *For a single-action, optimal auction, define the augmented (direct) inverse* $\breve{\psi}_i^{-1}$ *as* $\breve{\psi}_i^{-1}(v) = \{0\} \cup \psi_i^{-1}(v_{j<i})$. *Then for*

$$\mathcal{V}^{\psi}(v) = \left\{ \{v^{\rightsquigarrow i}\} : v^{\rightsquigarrow i} \in \breve{\psi}_i^{-1}(v), \sum_i \tilde{x}_i^*(v^{\rightsquigarrow i}) \leqslant 1 \right\},$$

*the auction is credible if and only if for all* $v$*, and* $\{v^{\rightsquigarrow i}\} \in \mathcal{V}^{\psi}(v)$,

$$\sum_i \tilde{t}_i(v^{\rightsquigarrow i}) \leqslant \sum_i \tilde{t}_i(v).$$

*Proof.* Apply Lemma D.1, specifically the unique mapping between $(\tilde{x}^*, \tilde{t})$ and $(x, t)$ to Definitions O.6 and O.7 to see that the lemma holds. □

**Lemma O.10.** *Suppose a single-action, optimal auction is weakly shill-proof, but not strongly shill-proof. Then, there exist* $R$*,* $v_R$*, and* $v_{-R}$ *such that*

$$\sum_{k \in R} \tilde{t}_k(v_R, v_{-R}) > \sum_{k \in R} \tilde{t}_k(v_R, 0). \tag{15}$$

*Proof.* Suppose that $(G, \sigma)$ is weakly shill-proof, but not strongly shill-proof. Because $(G, \sigma)$ is weakly shill-proof, for all $v$ and $R, R'$, we can define $\hat{\sigma}(v) \equiv \sigma(v; R) = \sigma(v; R')$. Since $(G, \sigma)$ is not strongly shill-proof, $\hat{\sigma}$ must not be an ex-post strategy for the shill bidders. So, for some realization of $R$ and $v_R$ there exists a profitable deviation for the shill bidders; examining the set of possible deviations $\Sigma_S$ in Definition A.1, we see that any profitable deviating actions induces a profitable misreport $v_{-R}$ in the direct mechanism for some $R$ and $v_R$; proving Equation (15) can be satisfied. □

### Proof of Proposition 6.2

**Weak Shill-Proofness → ($\psi$ = Id)-Credibility.** Suppose the auction is not ($\psi$ = Id)-credible. Then, combining Lemma O.9 with the ex-post IR condition, there exist $v, \{v^{\rightsquigarrow i}\} \in \mathcal{V}^{\psi}(v)$ and $k^*$ such that $\tilde{t}_{k^*}(v^{\rightsquigarrow k^*}) > \tilde{t}_{k^*}(v)$. Applying the winner-paying property, it is the case that for all $j \neq k^*$, $\tilde{t}_j(v^{\rightsquigarrow k^*}) = 0$. Since $\psi$ = Id, for all $j \leqslant k$, it is the case that $v_j^{\rightsquigarrow k^*} = v_j$ or $v_j = 0$. Let $R = \{1, \ldots, k^*\}$. Then,

$$\begin{aligned}
\sum_{i \in R} \tilde{t}_i(v^{\rightsquigarrow k^*}) &= \tilde{t}_{k^*}(v^{\rightsquigarrow k^*}) \\
&\geqslant \tilde{t}_{k^*}\left(v_1, \ldots, v_{k^*}, v_{k^*+1}^{\rightsquigarrow k^*}, \ldots, v_N^{\rightsquigarrow k^*}\right) \\
&\geqslant \tilde{t}_{k^*}\left(v_1, \ldots, v_{k^*}, 0, \ldots, 0\right) \\
&= \sum_{i \in R} \tilde{t}_i(v_R, 0).
\end{aligned}$$

Thus, we can apply Lemma D.2 to conclude that the auction is not weakly shill-proof.

**$\psi$-Credibility → Weak Shill-Proofness.** Suppose the auction is $\psi$-credible. Towards contradiction, suppose the auction is not weakly shill-proof. So, there exists $R$ and $v$ such that $\sigma(v; R) \neq \sigma(v; B)$. In particular, this means that shill bidders have, in expectation, a profitable deviation relative to acting as real bidders with valuation 0. If this is true in expectation, there must then exist $v = (v_R, 0)$ and $\tilde{v}_{-R}$ such that

$$\sum_{i \in R} \tilde{t}_i((v_R, \tilde{v}_{-R})) > \sum_{i \in R} \tilde{t}_i((v_R, 0)).$$

Now, let us consider the messaging deviation

$$\{v^{\rightsquigarrow i}\}_{i \in B} = \begin{cases} (v_R, \tilde{v}_{-R}) & i \in R \\ (v_R, 0) & \text{otherwise} \end{cases}.$$

By the definition of credibility, the auctioneer can report any value to other bidders when the value reported to him is 0 and bidders with value 0 are told the other bidders' true reports. Therefore, $\{v^{\rightsquigarrow i}\} \in \mathcal{V}^{\psi}(v_R, 0)$ and

$$\sum_i \tilde{t}_i(v^{\rightsquigarrow i}) = \sum_{i \in R} \tilde{t}_i(v_R, \tilde{v}_{-R}) + \sum_{i \notin R} \tilde{t}_i(v_R, 0) > \sum_i \tilde{t}_i((v_R, 0)).$$

This contradicts Lemma O.9, and so the auction must be weakly shill-proof.

**$(\psi = \varnothing)$-Credibility → Strong Shill-Proofness.** Suppose that the auction is not strongly shill-proof and $\psi = \varnothing$. There are two cases to consider: either the auction is not weakly shill-proof or it is. If the auction is not weakly shill-proof, then we can apply the previous case to conclude the auction is not $(\psi = \varnothing)$-credible. If the auction is weakly shill-proof, but not strongly shill-proof, then by Lemma O.10, there exists $R, k^* \in R, v_R$, and $v_{-R}$ such that $\tilde{t}_{k^*}(v) > \tilde{t}_{k^*}(v_R, 0)$. Thus, we can construct the following profitable auctioneer reporting deviation:

$$\{v^{\rightsquigarrow i}\}_{i \in B} = \begin{cases} (v_R, v_{-R}) & i = k^* \\ (v_R, 0) & \text{otherwise} \end{cases}.$$

Since $\psi = \varnothing$, we know that $\{\tilde{v}^{\rightsquigarrow i}\} \in \mathcal{V}^{\psi}(v_R, 0)$. The total transfers is then

$$\sum_i \tilde{t}_i\left(v^{\rightsquigarrow i}\right) = \tilde{t}_{k^*}(v) + \sum_{i \neq k^*} \tilde{t}_i\left(v_R, 0\right) > \sum_i \tilde{t}_i\left(v_R, 0\right);$$

hence, by Lemma O.9, we see that the auction is not credible.

**Strong Shill-Proofness → $\psi$-Credibility.** Suppose that the auction is not $\psi$-credible. Then, combining Lemma O.9 with the ex-post IR condition, there exists $v, \{v^{\rightsquigarrow i}\} \in \mathcal{V}^{\psi}(v)$ and $k^*$ such that $\tilde{t}_{k^*}(v^{\rightsquigarrow k^*}) > \tilde{t}_{k^*}(v)$. Recall, by the definition of $\psi^{-1}$, that $v_{k^*}^{\rightsquigarrow k^*} = v_{k^*}$. Suppose $R = \{k^*\}$. Then,

$$\sum_{i \in R} \tilde{t}_i(v^{\rightsquigarrow k^*}) = \tilde{t}_{k^*}(v^{\rightsquigarrow k^*}) > \tilde{t}_{k^*}(v) \geqslant \tilde{t}_{k^*}(v_{k^*}, 0) = \sum_{i \in R} \tilde{t}_{k^*}(v_{k^*}, 0).$$

Therefore, by Lemma D.3, the auction is not strongly shill-proof. □